Part I - Optimization (IEOR 240 content)

All the answers have to be clearly justified. An answer with no justification will lead to no score for the question.

Exercise 1

A renewable energy company plans to expand its electricity production capacity by at least 2500 megawatts (MW) to meet growing energy demands. The company wants to minimize costs and has identified several sites where new plants can be built. Each site has a base capacity that comes with a fixed cost, and some sites offer the option to expand capacity further with additional investments. The data for the potential sites is as follows:

Site	Base Cost	Base Capacity	Additions Possible
1	\$9M	750 MW	None
2	\$6M	450 MW	None
3	\$13M	1100 MW	\$0.015M per MW, up to +350 MW
4	\$9M	800 MW	\$0.02M per MW, up to +400 MW

Further, the contractor we are working with has agreed to give discount on building (the base version of) site 4 if sites 1 and 2 are built: If (the base versions of) sites 1 and 2 are built, then site 4 can be built for \$7M instead of \$9M (but the additions cost the same).

Formulate the problem of expanding the company's electricity production by at least 2500 MW at minimum cost as a Mixed-Integer Linear Program (MILP). Clearly define your decision variables, objective function, and constraints, and describe the purpose of each constraint in a few words.

Exercise 2

A company is making chairs and tables, and has the following technology table that summarizes their available resources in the rows, and the resources needed to produce chairs and tables as the columns:

	Chair	Table	Units available
Labor	3	2	130
Wood	1	2	80
Paint	0.4	0.8	45

a) The company's client is willing to buy any number of chairs for $25, and any number of tables for $35. Formulate as an LP the problem of deciding how many chairs and tables to produce and sell to the client, subject to the resource information provided in the table, to maximize the company's revenue.

b) Suppose that you solve the LP and find that the shadow price of wood is 55, and that the shadow price of paint is 0. Explain what that tells you, and what decisions the company might want to use that information for.

c) Suppose instead that the company's client is willing to buy tables for \$35 per table for up 20 tables, and \$25 for up to 15 additional tables beyond that. The client is still willing to buy chairs for \$25 per chair, but only up to 6 chairs per table purchased.

Further, if the company makes 20 or more tables, they would like to make at least 10 chairs so that their staff doesn't forget how to make chairs.

Given this new setting, formulate as an ILP or MILP the optimization problem of maximizing the revenue of the company. Describe in a few words any *new* decision variable, modification to the objective function, or new constraint you have introduced.

Exercise 3

For each of the following statements, determine whether it is true or false. **Justify your answer.**

a) An LP is always either unbounded, infeasible, or has a unique optimal solution.

b) When a slack variable for a resource constraint in an LP is positive in an optimal solution, the shadow price for that resource must be 0.

c) Let u_A, u_B, u_C be binary decision variables for yes/no decisions A, B, and C, and let x be a continuous decision variable with $x \geq 0$, $x \leq 100$. The following constraint can be captured by linear constraints without introducing additional decision variables: if we decide "no" for decision B, then we must have $x \leq 50$ and we must decide "yes" on at least one of A or C.

Part II - Stochastic Modeling (IEOR 241 content)

All the answers have to be clearly justified. An answer with no justification will lead to no score for the question.

4 exercises with one question + 2 exercises with 2 questions

Exercise 1

Two fair (cube) dice have had two of their sides painted red, two painted black, one painted yellow and the last one painted in blue. **Question:** When this pair of fair dice is rolled, what is the probability that both dice land with the same color face up?

Exercise 2

An insurance company classifies people into three classes: good risks, average risks, bad risks. The company records indicate that the probability that good, average or bad-risk persons will be involved in at least one accident over a 1-year span are $0.05, 0.15, 0.30$ respectively. We estimate that 10% of the population is good-risk, 60% are bad risk and 30% are bad risk. **Question:** If a policyholder has no accident in 2024, what is the probability that the policyholder is a good-risk?

Exercise 3

An electric system is composed with n transistors, each of which will, independently, functions with probability p. The total system will be able to operate effectively if at least one-half of its components function. **Question:** Give a condition on p ensuring that a system with 5 transistor will be more likely to operate effectively than a 3 transistor system.

Exercise 4

Ten percent of the population is left-handed. There are currently 219 students enrolled in the course INDENG 241. **Question:** Approximate the probability that the number of students in INDENG 241 who are left-handed is between 15 and 25 (inclusive) by using DeMoivre Laplace theorem with a correction of continuity at the threshold 0.5, that is to find $\mathbb{P}(14.5 \leq X \leq 25.5)$ for X a random variable to be determined denoting the number of student left handed in 241.

Exercise 5

A city launch a bikeshare program with 3 bike stations, Station 0, Station 1, and Station 2. People can borrow a bicycle at one station and return it to the same station or either of the other two stations. We assume that if a bike is borrowed at the Station 0, it will return at Station 0 with probability 0.5, it will be return to station 1 with probability 0.4 and return to Station 2 with probability 0.1. If the bike is borrowed at the station 1, it will return to either station 0 or Station 1 with equal probability 0.5. If a bike is borrowed at the station 2, it will be return at the station 1 with probability 1.

Ex. 5-Question 1: Draw the graph and give the transition matrix P of this Markov chain.

Ex. 5-Question 2: What is the proportion of time we will find a bike at the station 0? That is find π_0 where $\pi = (\pi_0, \pi_1, \pi_2)$ solves $\pi P = \pi$.

Exercise 6

A store stocks two different brands of a certain type of flour. Let X the amount (in lb) of brand A and Y the amount of brand B (in lb)). We suppose the joint density of X and Y is

$$f(x, y) = \begin{cases} \alpha x y, \text{ if } 0 \leq x \leq 10,\ 0 \leq y \leq 20,\ x \leq y \\ 0 \text{ otherwise.} \end{cases}$$

Ex. 6-Question 1: Find the value of α

Ex. 6-Question 2: Compute the probability that the total stock of flour is less than 15 lbs (brand A plus brand B, that is $\mathbb{P}(X + Y \leq 15)$).

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF X

X	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998