Sample IEOR Technical Comprehensive Exam Questions - Stochastic Modeling

1. A and B are two events. Suppose $P_r(A) = 0.3, P_r(A \cup B) = 0.8$. Let $P_r(B) = p$ with p to be determined.

   (a) For what choice of p is $P_r(A \cap B) = 0$?

   (b) For what choice of p, A and B are independent?

2. Let G, F be two arbitrary events. The following statement is true or false? $P(G|G \cup F) > P(G|F)$. Prove why it is true, or give an example to show why it is false.

3. Let P_n denote the probability that in n tosses of a fair coin, no run of 2 consecutive heads appears.

   (a) Total) Write down the recursive equation for P_n in terms of P_{n-1}, P_{n-2}.

   (b) Find P_1, P_2, P_3, P_4, P_5.

4. IEOR Department has 11 male professors and 4 female professors.

   (a) A committee consisting 3 men and 2 woman is to be selected from the department. How many distinct committees are possible? What is the probability that a particular woman professor, say Professor Guo, is part of the committee member?

   (b) A committee consisting 3 men and 2 woman is to be selected from the department with one of the committee members designated as chairman. How many distinct committees are possible? What is the probability that one particular male professor, say Professor Ilan Adler, is the chair of the committee?

5. Suppose e-mails arrive in your computer account at an average rate of 5 for every hour. Assume the arrival time between e-mails is exponentially distributed. You checked your e-mail just before coming to the exam (6 pm). What is the probability you have not received any new messages by the time you check your e-mail again right after the exam (at 9 pm)?

6. In answering a question in a multiple choice test, a student either knows the answer or guesses. Suppose 90 percent of the time she knows the correct answer and 10 percent of the time she guesses. Assume that if the student knows the correct answer, she will always answer the question correctly and if she guesses, she will answer correctly with probability 0.4. Given that a student answers the question correctly, what is the probability that she actually knows the answer?

7. Suppose $f(x)$ is a probability density function, given by $f(x) = cxe^x$ for $0 < x < 1$, for some constant c.

(3a) Determine c.

(3b) Find $Var[(2+X)]$.

8. Going to a casino and playing a slot machine of a particular game. The chance of winning in each round is 0.4. Alex decides that he will play 100 rounds. What is the probability that in this 100 round, he wins at least 45 times? Write down the expression and approximate it.

9. Suppose that X and Y have joint density function $f(x,y) = \frac{1}{x^2y^2}$ for $x \geq 1, y \geq 1$.

   (a) Compute the joint density function of (U,V) with $U = 2X, V = \frac{Y}{2X}$.

   (b) Find the marginal densities for U and V.

   (c) Are U and V independent? Why or Why not?

10. A coin is flipped with probability p of being head (H) and $1-p$ of being tail (T). Find $E[N]$ where N is the number of flips needed to see pattern THH.

11. Suppose that a particular gene occurs as one of two alleles (A and a) where allele A has frequency θ in the population. That is, a random copy of the gene is A with probability θ and a with probability $1-\theta$. Since a diploid genotype consists of two genes, the probability of each genotype is given by

genotpye	AA	Aa	aa
probability	θ^2	$2\theta(1-\theta)$	$(1-\theta)^2$

   Suppose we test a random sample of people and find that k_1 are AA, k_2 are Aa and k_3 are aa. Find the MLE of θ.

12. Judge if the following statement is true or false, and give your reasoning. If the statement is false, give an example to show why the statement does not hold. If the statement is true, explain why it is true.

    "Given any random variables X, Y, Z. If X is independent of Y, Y is independent of Z, X is independent of Z, then X, Y, Z are mutually independent."

13. Let $N(T)$ be a Poisson process with $\lambda = 0.4$, and let the inter-arrival times be $T_1, T_2, \cdots$, and $S_n = T_1 + \cdots T_n$.

    2a) (5 points) What is the distribution of S_5?

    2b) (5 points) Compute $E[S_3|S_5 = 4]$

14. There are two servers in a queuing system to process requests, where the service time of servers $i (i = 1,2)$ is exponentially distributed with rate μ_i. Whenever a server becomes free, the customer who has been waiting the longest begins the service with that server. If you arrive to find out all servers are busy and one person waiting, find the expected time until you depart the system.

15. (10 points) A test of detecting cancer has been developed. Suppose that 99 percent of the patients having cancer reacted positively to the test, and only 2 percent of the those not having cancer reacted positively to the test. Suppose that 2 percent of the patient in the hospital have cancer. What is the probability that a patient selected at random, who reacts positively to the test, will actually have cancer?