Region partitioning heuristic

1. Rectangle containing customers,
2. Partition so that there are \( q \) customers in each region,
3. Find optimal TSP in each region,
4. Connect these to find a tour.

An \( O(q^22^q) \) forward DP based algorithm for the TSP

\(-\text{Held and Karp(1962)}\)

Notations

\( S \): set of cities.

\( S_l \): an \( l-1 \) city subset of \( S\setminus\{1\} \).

\( cost(S_l, i) \): the minimum cost of an \( l \)-city path starting at 1 and ending at \( i \) for a particular \( S_l \).

\[
\text{cost}(S^l, i) = \min_{k \in S^l \setminus \{i\}} \{\text{cost}(S^l \setminus \{i\}, k) + c_{ki}\}.
\]

Algorithm

for \( i = 1, 2, \ldots, n \) \( \text{cost}(\{i\}, 1) = c_{1i}, \text{bestpath}(\{i\}, 1) = (1, i) \).

\[
C \left\{ \begin{array}{l}
\text{for } j = 2, 3, \ldots, n \\
\quad \text{for each } S^j \\
\quad \quad \text{for each } i \in S^j \\
\quad \quad \quad \text{cost}(S^j, i) = \min_{k \in S^j \setminus \{i\}} \{\text{cost}(S^j \setminus \{i\}, k) + c_{ki}\} \\
\quad \quad k \text{ is the city that achieves the minimum} \\
\quad \quad \text{bestpath}(S^j, i) = \text{bestpath}(S^j \setminus \{i\}, k) + \{i\}
\end{array} \right.
\]
mincost = min_{k\neq 1} \{ \text{cost}(S^{n-1}\setminus\{i\}, k) + c_k \}

k the city that achieves this.

bestpath = bestpath(S^{n-1}, k) + \{i\}.

**Complexity**

A : for each \( i \in S^j \), \( j \) times \( \rightarrow O(j^2) \),

B : \( \binom{n-1}{j-1} \) possible \( S^j \),

C : for all \( j \), \( \sum_{j=2}^{n} \left( \frac{n-1}{j-1} \right) j^2 \).

This is not polynomial but if \( q = \sqrt{\log n} \) then the running time for a subregion is \( O(n \log n^2) \). And since there are at most \( \frac{n}{\log n} + 1 \) subregions, the running time to solve all the TSPs is \( O\left( \frac{n}{\log n} \log n^2 \right) = O(n^2 \log n) \).

**The region partitioning**

Given a rectangle with width \( a \) and length \( b \) containing all the points, subdivide the rectangle into regions containing \( q \) customers.

Divide the region with \( t \) vertical lines and \( h \) horizontal lines.

Index customers in order of horizontal coordinate.

Place the \( j \)-th vertical line, \( j = 1, 2, \ldots, t \) through customer with index
$j(k + 1)q$.
This leads to $t + 1$ vertical strips each with $(h + 1)q$ customers.
Now use $h$ horizontal lines to partition each strip into $h + 1$ smaller subregions.

Each subregion has $q$ customers, except possibly the last one (counting right, bottom).
Now solve TSP for each subregion (up to $q + 2$ point in interior regions).
This is an eulerian graph, so convert the TSP tour using shortcuts.

**Bounds**

The following hold:

\[
\begin{align*}
t &= \left\lceil \frac{n}{(h + 1)q} \right\rceil - 1 \leq \sqrt{\frac{n}{q}} \\
t(h + 1)q &< n \leq (t + 1)(h + 1) \\
h &= \left\lceil \sqrt{\frac{n}{q}} - 1 \right\rceil \leq \sqrt{\frac{n}{q}}
\end{align*}
\]

**Theorem**:

If $X_i$ are i.i.d. random variables with compact support in $\mathbb{R}^2$, then:

\[
\lim_{n \to \infty} \frac{L^*}{\sqrt{n}} = \lim_{n \to \infty} \frac{L^{RP}}{\sqrt{n}}.
\]

To prove this, we need the following lemma (we will apply it this class and prove it next class).

**Lemma**:

\[
L^* \leq L^{RP} \leq L^* + \frac{3}{2}p^{RP}
\]

where $p^{RP}$ is the sum of perimeters of subregions.

**Proof of the theorem**

- added vertical lines $t \leq \sqrt{\frac{n}{q}}$, counted twice, length $a$, 

– added horizontal lines $h \leq \sqrt{\frac{n}{q}}$, counted twice, length $b$,
– $a$, $b$ counted twice.
So :

$$p^{RP} \leq 2\sqrt{\frac{n}{q}}(a + b) + 2(a + b).$$

Since $q = \lceil \log n \rceil$ :

$$p^{RP} \leq 2\sqrt{\frac{n}{\log n}}(a + b) + 2(a + b).$$

Applying the lemma :

$$L^* \leq L^{RP} \leq \frac{L^*}{\sqrt{n}} + \frac{3p^{RP}}{\sqrt{n}} \leq \frac{L^*}{\sqrt{n}} + 3(a + b)(\frac{1}{\sqrt{\log n}} + \frac{1}{\sqrt{n}}).$$

Taking the limit $n \to \infty$ completes the proof.