NP-Completeness

3-Satisfiability

Given a set of literals \((X_1, X_2, ..., X_n)\) and an expression \(F = C_1 \cdot C_2 \cdot ... \cdot C_m\) such that \(|C_i| = 3\) for all \(i\), is there an assignment of labels to literals such that every clause is satisfied?

**Theorem:** 3-SAT is NP-complete.

**Sketch of Proof:**

1. In NP.
2. Show SAT polynomially transforms to 3-SAT.
   (i.e. Given an expression that is satisfiable, construct an expression consisting of clauses with 3 literals that is satisfiable if and only if the initial expression is satisfiable.)

Consider an instance of SAT with \(X_1, X_2, ..., X_n\) literals.

\[ F = C_1 \cdot C_2 \cdot ... \cdot C_m \]

Consider a new instance

\[ F' = C'_1 \cdot C'_2 \cdot ... \cdot C'_m \]

such that \(|C'_i| = 3\) and \(F'\) is satisfiable if and only if \(F\) is satisfiable.

Replace each \(C_j\) by a number of new clauses utilising the original literals and some new ones: \(Y^1_j, Y^2_j\), etc.

If \(|C_j| = 1\), \(C_j = X_i\)

\[ C'_j = (X_i + Y^1_j + Y^2_j) \cdot (X_i + \overline{Y}^1_j + Y^2_j) \cdot (X_i + Y^1_j + \overline{Y}^2_j) \cdot (X_i + \overline{Y}^1_j + \overline{Y}^2_j) \]

If \(|C_j| = 2\), \(C_j = X_i + X_2\)

\[ C'_j = (X_i + X_2 + Y^1_j) \cdot (X_i + X_2 + \overline{Y}^1_j) \]

If \(|C_j| > 3\), \(C_j = X_i, X_2, ..., X_L\)
\[ C'_j = (X_1 + X_2 + Y^j_1) \cdot (\overline{Y^j_1} + X_3 + Y^2_1) \cdot (\overline{Y^2_1} + X_4 + Y^3_1) \cdot \ldots \cdot (\overline{Y^{L-3}_1} + X_{L-2} + Y^{L-3}_1) \cdot (\overline{Y^{L-3}_1} + X_{L-1} + X_L) \]

For the proof, we need to show that \( F' \) is satisfiable if and only if \( F \) is satisfiable.

First 2 cases are easy; final case:

Let \( l \) be minimizing \( i \) such that \( X_i \) is true in \( F \).

If \( l = 1, 2, \) \( Y^j_i = \text{FALSE} \), \( F' \) is satisfied.

If \( l = L - 1, L - 2, \) \( Y^j_i = \text{TRUE} \), \( F' \) is satisfied.

Otherwise, set \( Y^j_i = \text{TRUE} \) for all \( i \) up to \( l - 2 \), and \( \text{FALSE} \) for the rest.

**Vertex Cover**

Given a graph \( G (V,E) \) and a positive integer \( K \leq |V| \), is there a vertex cover of size \( K \) or less?

Equivalently, is there \( V' \leq V \) with \(|V| \leq K \) such that \((i,j) \in E\), either \( i \) or \( j \) or both are in \( V' \)?

**Example:**

<table>
<thead>
<tr>
<th>( V' ) = 2, select 1 &amp; 3 for VC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
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<tr>
<td>4</td>
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</tbody>
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**Theorem:** VC is NP-complete.

**Sketch of Proof:**

1. Clearly, VC is in NP.
2. Show that 3-SAT polynomially transforms to VC.
   (i.e. Given an instance of 3-SAT, construct an instance of VC such that solution to one is yes if and only if solution the other is yes.)
Consider the literals \( X_1, X_2, \ldots, X_n \) (\( n \) terms) and the expression \( F = C_1 \cdot C_2 \cdot \ldots \cdot C_m \) (\( m \) terms) where \( |C_j| = 3 \). We construct a graph \( G(V,E) \) and define a positive integer \( K \) such that \( G \) has a vertex cover of size \( \leq K \) if and only if \( F \) is satisfied.

**Constructing the graph:**

- For every literal \( X_i \), we build 2 nodes \( X_i, \overline{X_i} \) and connect them with an edge.
- For every clause \( C_j \), construct 3 nodes \( a_j^1, a_j^2, a_j^3 \), and connect them to form a triangle.
- For every clause \( C_j \), \( C_j = \{ \alpha_j + \beta_j + \gamma_j \} \), connect \( a_j^1 - \alpha_j, a_j^2 - \beta_j, a_j^3 - \gamma_j \).

**Example:** \( F = (X_1 + \overline{X}_3 + \overline{X}_4) \cdot (\overline{X}_1 + X_2 + \overline{X}_4) \)

Now, let \( K = n + 2m \).

First, we show that \( \text{VC} \) “Yes” \( \rightarrow \) \( 3\text{-SAT} \) “Yes”.

- Let \( V' \) be the VC with \( |V'| \leq K \).
- \( V' \) must contain at least one vertex from each \( X_i, \overline{X}_i \) pair (since each pair is disconnected).
- \( V' \) must contain at least 2 vertices from each triangle (to cover all 3 arcs in the triangle).

\[
\begin{align*}
 n + 2m & \geq |V'| \geq n + 2m \\
 \Rightarrow |V'| &= n + 2m 
\end{align*}
\]

Thus, \( \text{VC} \) has exactly one of each \( X_i, \overline{X}_i \) pair and 2 from each triangle.

**Claim:** If \( X_i \in V' \), \( X_i \) is TRUE

\( \overline{X}_i \in V' \), \( \overline{X}_i \) is TRUE
is the 3-SAT solution set.

To see that this is true, consider the edges $a_i^1 - \alpha_i, a_i^2 - \beta_i, a_i^3 - \gamma_i$. Since only 2 of $a_i^1, a_i^2, a_i^3$ are in $V'$, one of the arcs is covered by a vertex that represents a literal that is true. So the clause is true. ■

Next, show that 3-SAT “Yes” $\rightarrow$ VC “Yes”.

If $F$ is satisfiable, construct $V' \subseteq V$ with $|V'| \leq K$ such that

If $X_i$ is TRUE, $X_i \in V'$.

If $X_i$ is TRUE, $\overline{X_i} \in V'$.

Therefore all “literal” arcs are covered, and one “critical-clause” arc from each clause is covered. Pick 2 other nodes in each triangle to cover the rest. ■

**Hamiltonian Cycle Problem**

Given graph $G (N,A)$, does $G$ contain a cycle that visits each node exactly once?

1. Clearly in NP.
2. VC polynomially transforms to HCP.

**TSP-Recog-I**

1. We have seen that it is in NP.
2. Show that HCP polynomially transforms to TSP.

**Sketch of Proof:**

Given a HCP, create a complete graph with the same nodes as HCP. For each arc in TSP, if it is in HCP, set cost = 1. Otherwise, set cost = 2.

$$B = |N|$$

- If a TSP solution with cost $\leq N$ exists, it only uses arcs in HCP, since it only uses arcs of cost 1.
- If HCP exists, it is also a TSP tour in the new graph with cost $\leq |N|$. 