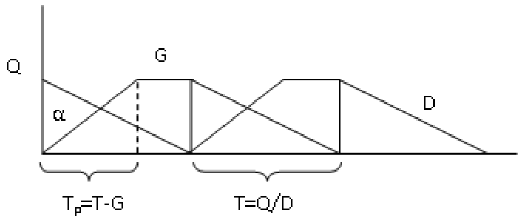
IEOR 250 Midterm Suggested Solution

1 a) Feasibility Condition: DG<C and $\alpha \geq Q/T_p$. Since we would like to minimize total cost, we set $\alpha = Q/T_p$



If you interpret the question as you will pay the leasing cost for the whole period, then the following formulation is correct.

$$\begin{aligned}
\alpha &= \frac{Q}{T_p} = \frac{Q}{T-G} = \frac{Q}{\frac{Q}{D}-G} = \frac{QD}{Q-GD} \\
\frac{Cost}{Cycle} &= C_1 + \alpha C_2 T + \frac{1}{2}hQT \\
\min \quad & TC(Q) = \frac{C_1 D}{Q} + C_2 \frac{QD}{Q-GD}\frac{Q}{D} + \frac{1}{2}hQ \\
\frac{\partial TC(Q)}{\partial Q} &= 0 = \frac{-C_1 D}{Q^2} + \frac{C_2 Q^2 - 2C_2 QGD}{(Q-GD)^2} + \frac{1}{2}h \Rightarrow Q^*
\end{aligned}$$

If you interpret the question as you only need to pay the leasing cost while using it, then the following formulation is also correct.

$$\begin{aligned}
\alpha &= \frac{Q}{T_p} = \frac{Q}{T-G} = \frac{Q}{\frac{Q}{D}-G} = \frac{QD}{Q-GD} \\
\frac{Cost}{Cycle} &= C_1 + \alpha C_2 (T-G) + \frac{1}{2}hQT \\
\min \quad & TC(Q) = \frac{C_1 D}{Q} + C_2 D + \frac{1}{2}hQ \\
\frac{\partial TC(Q)}{\partial Q} &= 0 = \frac{-C_1 D}{Q^2} + \frac{1}{2}h \Rightarrow Q^* = \sqrt{\frac{2C_1 D}{h}}
\end{aligned}$$

Production Strategy:
If $Q^* > C$, start cooking whenever there are C sausages extruded.
If $DG \le Q^* \le C$, start cooking whenever there are Q^* sausages extruded.
If $Q^* < DG$, start cooking whenever there are DG sausages extruded.

1 b) We need $\alpha \geq$ D to ensure feasibility and we would like to minimize total cost, so we set α=D.

$$\min\ TC(Q) = \frac{C_1 D}{Q} + C_2 D + \frac{1}{2}hQ = C_2 D + \frac{1}{2}hQ$$

Total cost per unit time is a strictly increasing function of Q, thus the optimal strategy is to start cooking whenever there are DG sausages extruded.

2 a)

Sell finished goods first

$$\begin{aligned} E[TC(f,r)] &= c_f f + c_r r - P\int_0^\infty \min(f+r, D)f(D)dD + c_t\int_f^\infty \min(D-f, r)f(D)dD \\ &= c_f f + c_r r - P\int_0^{f+r} Df(D)dD - P\int_{f+r}^\infty (f+r)f(D)dD \\ &\quad + c_t\int_f^{f+r}(D-f)f(D)dD + c_t\int_{f+r}^\infty rf(D)dD \end{aligned}$$

Sell goods that are transformed from raw materials first

$$\begin{aligned} E[TC(f,r)] &= c_f f + c_r r - (P-c_t)\int_0^\infty \min(D,r)f(D)dD + P\int_r^\infty \min(D-r, f)f(D)dD \\ &= c_f f + c_r r - (P-c_t)\int_0^r Df(D)dD - (P-c_t)\int_r^\infty rf(D)dD \\ &\quad + P\int_r^{r+f}(D-r)f(D)dD + Pf\int_{r+f}^\infty f(D)dD \end{aligned}$$

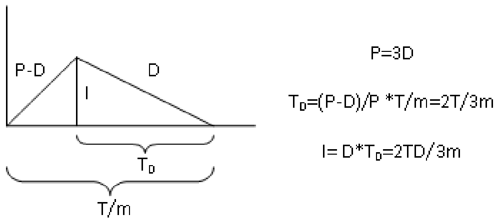
2 b) It is cheaper to buy raw materials and transform it into finished goods as needed, thus f^*=0. This problem then becomes the newsvendor problem we studied in class.

$$\begin{aligned} E[TC(r)] &= c_r r - (P-c_t)\int_0^r Df(D)dD - (P-c_t)\int_r^\infty rf(D)dD \\ \frac{\partial E[TC(r)]}{\partial r} &= c_r - (P-c_t)\int_r^\infty f(D)dD = 0 \\ &\Rightarrow\ Pr\{D \le r\} = \frac{P-c_t-c_r}{P-c_t} \end{aligned}$$

2 c) Setting r=0 is not necessarily optimal. Suppose $c_r < c_f$ and the demand D has a large variance, i.e. there is a possibility for the demand to be really small. It is then better to purchase some raw materials and transform them into goods only when there is the demand for it.

$$\text{Set } \frac{\partial E[TC(f,r)]}{\partial f} = 0 \text{ and } \frac{\partial E[TC(f,r)]}{\partial r} = 0 \Rightarrow r^*, f^*$$

3 a)



Let m_i be the number of orders placed for product i in the interval $[0, T]$. Let $V_i(t)$ be the inventory level for product i at time t, maximum capacity $\geq$ average capacity over this period is

$$\frac{1}{T}\int_0^T \sum_i \alpha_i V_i(t) \geq \frac{1}{T}\sum_{i=1}^n \alpha_i \frac{T}{m_i}\frac{2T}{3m_i}\frac{D_i}{2} m_i = \frac{1}{3}\sum_{i=1}^n \alpha_i \frac{TD_i}{m_i} = \frac{1}{3}\sum_{i=1}^n \alpha_i Q_i$$

$$\begin{aligned} TC &= \sum_i \{\frac{K_i m_i}{T} + \frac{h_i}{T}\int_0^T V_i(t)dt\} \geq \sum_i \{\frac{K_i m_i}{T} + \frac{h_i}{T}\frac{T}{m_i}\frac{2TD_i}{3m_i}\frac{m_i}{2}\} \\ &= \sum_i \{\frac{K_i m_i}{T} + \frac{h_i D_i T}{3m_i}\} = \sum_i \{\frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{3}\} \end{aligned}$$

$$\begin{aligned} Z^{LB} = \min \quad & \sum_i \frac{K_i D_i}{Q_i} + \frac{h_i I_i}{2} = \sum_i \frac{K_i D_i}{Q_i} + \frac{h_i}{2}\frac{Q_i(P_i - D_i)}{P_i} = \sum_i \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{3} \\ & \frac{1}{3}\sum_i \alpha_i Q_i \leq B \\ & Q_i \geq 0 \ \forall i \end{aligned}$$

3 b)

$$\begin{aligned} Z^{H} = \min \quad & \sum_i \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{3} \\ & \sum_i \alpha_i I_i \leq B \Rightarrow \frac{2}{3}\sum_i \alpha_i Q_i \leq B \\ & Q_i \geq 0 \ \forall i \end{aligned}$$

3 c) Let Q_i^{LB} be the solution to Z^{LB},and $Q_i = \frac{Q_i^{LB}}{2}$ is feasible to Z^H.

$$\begin{aligned}
Z^H &\leq \sum_i \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{3} = \sum_i \frac{2K_i D_i}{Q_i^{LB}} + \frac{h_i Q_i^{LB}}{6} \\
&\leq \sum_i \frac{2K_i D_i}{Q_i^{LB}} + 2\frac{h_i Q_i^{LB}}{3} = 2Z^{LB} \\
\frac{Z^H}{Z^*} &\leq \frac{Z^H}{Z^{LB}} \leq 2
\end{aligned}$$