IEOR 250 Midterm Suggested Solution

1 a) Feasibility Condition: \( DG < C \) and \( \alpha \geq Q/T_p \). Since we would like to minimize total cost, we set \( \alpha = Q/T_p \).

\[ \begin{align*}
\alpha &= \frac{Q}{T_p} = \frac{Q}{T - G} = \frac{Q}{D - G} = \frac{QD}{Q - GD} \\
\text{Cost} \quad \frac{\text{Cycle}}{\text{Cost}} &= C_1 + \alpha C_2 (T - G) + \frac{1}{2} h QT \\
\text{min} \quad TC(Q) &= C_1 D + \frac{C_2 QD}{Q - GD} + \frac{1}{2} h Q \\
\frac{\partial TC(Q)}{\partial Q} &= 0 = \frac{-C_1 D}{Q^2} + \frac{C_2 Q^2 - 2C_2 QGD}{(Q - GD)^2} + \frac{1}{2} h \Rightarrow Q^* \\
\end{align*} \]

If you interpret the question as you will pay the leasing cost for the whole period, then the following formulation is correct.

\[ \begin{align*}
\alpha &= \frac{Q}{T_p} = \frac{Q}{T - G} = \frac{Q}{D - G} = \frac{QD}{Q - GD} \\
\text{Cost} \quad \frac{\text{Cycle}}{\text{Cost}} &= C_1 + \alpha C_2 (T - G) + \frac{1}{2} h QT \\
\text{min} \quad TC(Q) &= C_1 D + \frac{C_2 QD}{Q - GD} + \frac{1}{2} h Q \\
\frac{\partial TC(Q)}{\partial Q} &= 0 = \frac{-C_1 D}{Q^2} + \frac{C_2 Q^2 - 2C_2 QGD}{(Q - GD)^2} + \frac{1}{2} h \Rightarrow Q^* \\
\end{align*} \]

If you interpret the question as you only need to pay the leasing cost while using it, then the following formulation is also correct.

\[ \begin{align*}
\alpha &= \frac{Q}{T_p} = \frac{Q}{T - G} = \frac{Q}{D - G} = \frac{QD}{Q - GD} \\
\text{Cost} \quad \frac{\text{Cycle}}{\text{Cost}} &= C_1 + \alpha C_2 (T - G) + \frac{1}{2} h QT \\
\text{min} \quad TC(Q) &= C_1 D + \frac{C_2 QD}{Q - GD} + \frac{1}{2} h Q \\
\frac{\partial TC(Q)}{\partial Q} &= 0 = \frac{-C_1 D}{Q^2} + \frac{1}{2} h \Rightarrow Q^* = \sqrt{\frac{2C_1 D}{h}} \\
\end{align*} \]
Production Strategy:
If $Q^* > C$, start cooking whenever there are $C$ sausages extruded.
If $DG \leq Q^* \leq C$, start cooking whenever there are $Q^*$ sausages extruded.
If $Q^* < DG$, start cooking whenever there are $DG$ sausages extruded.

1 b) We need $\alpha \geq D$ to ensure feasibility and we would like to minimize total cost, so we set $\alpha = D$.

$$\min TC(Q) = \frac{C_1 D}{Q} + C_2 D + \frac{1}{2} hQ = C_2 D + \frac{1}{2} hQ$$

Total cost per unit time is a strictly increasing function of $Q$, thus the optimal strategy is to start cooking whenever there are $DG$ sausages extruded.

2 a) Sell finished goods first

$$E[TC(f, r)] = c_f f + c_r r - P \int_0^\infty \min(f + r, D)f(D)dD + c_t \int_f^{f+r} \min(D - f, r)f(D)dD$$

$$= c_f f + c_r r - P \int_0^{f+r} Df(D)dD - P \int_{f+r}^\infty (f + r)f(D)dD$$

$$+ c_t \int_f^{f+r} (D - f)f(D)dD + c_t \int_{f+r}^\infty r f(D)dD$$

Sell goods that are transformed from raw materials first

$$E[TC(f, r)] = c_f f + c_r r - (P - c_t) \int_0^{\infty} \min(D, r)f(D)dD + P \int_r^{\infty} \min(D - r, f)f(D)dD$$

$$= c_f f + c_r r - (P - c_t) \int_r^{\infty} Df(D)dD - (P - c_t) \int_r^{\infty} r f(D)dD$$

$$+ P \int_r^{r+f} (D - r)f(D)dD + Pf \int_{r+f}^\infty f(D)dD$$

2 b) It is cheaper to buy raw materials and transform it into finished goods as needed, thus $f^* = 0$. This problem then becomes the newsvendor problem we studied in class.

$$E[TC(r)] = c_r r - (P - c_t) \int_r^{\infty} Df(D)dD - (P - c_t) \int_r^{\infty} r f(D)dD$$

$$\frac{\partial E[TC(r)]}{\partial r} = c_r - (P - c_t) \int_r^{\infty} f(D)dD = 0$$

$$\Rightarrow Pr\{D \leq r\} = \frac{P - c_t - c_r}{P - c_t}$$
2) Setting \( r=0 \) is not necessarily optimal. Suppose \( c_r < c_f \) and the demand \( D \) has a large variance, i.e. there is a possibility for the demand to be really small. It is then better to purchase some raw materials and transform them into goods only when there is the demand for it.

Set \( \frac{\partial E[TC(f, r)]}{\partial f} = 0 \) and \( \frac{\partial E[TC(f, r)]}{\partial r} = 0 \Rightarrow r^*, f^* \)

3) a)

Let \( m_i \) be the number of orders placed for product \( i \) in the interval \([0, T]\). Let \( V_i(t) \) be the inventory level for product \( i \) at time \( t \), maximum capacity \( \geq \) average capacity over this period is

\[
\frac{1}{T} \int_0^T \sum_i \alpha_i V_i(t) dt \geq \frac{1}{T} \sum_{i=1}^n \alpha_i \frac{T}{3} \frac{2T D_i}{m_i} \frac{m_i}{2} = \frac{1}{3} \sum_{i=1}^n \alpha_i \frac{T D_i}{m_i} = \frac{1}{3} \sum_{i=1}^n \alpha_i Q_i
\]

\[
TC = \sum_i \left( \frac{K_i m_i}{T} + \frac{h_i}{T} \int_0^T V_i(t) dt \right) \geq \sum_i \left( \frac{K_i m_i}{T} + \frac{h_i}{T} \frac{2T D_i m_i}{m_i} \right)
\]

\[
Z^{LB} = \min \sum_i \frac{K_i D_i}{Q_i} + \frac{h_i I_i}{2} = \sum_i \frac{K_i D_i}{Q_i} + \frac{h_i}{2} \frac{Q_i (P_i - D_i)}{P_i} = \sum_i \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{3}
\]

\[
\frac{1}{3} \sum_i \alpha_i Q_i \leq B
\]

\[
Q_i \geq 0 \ \forall i
\]

3) b)

\[
Z^H = \min \sum_i \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{3}
\]

\[
\sum_i \alpha_i I_i \leq B \Rightarrow \frac{2}{3} \sum_i \alpha_i Q_i \leq B
\]

\[
Q_i \geq 0 \ \forall i
\]
3 c) Let $Q_i^{LB}$ be the solution to $Z^{LB}$, and $Q_i = \frac{Q_i^{LB}}{2}$ is feasible to $Z^H$. 

\[ Z^H \leq \sum_i \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{3} = \sum_i \frac{2K_i D_i}{Q_i^{LB}} + \frac{h_i Q_i^{LB}}{6} \]

\[ \leq \sum_i \frac{2K_i D_i}{Q_i^{LB}} + \frac{h_i Q_i^{LB}}{3} = 2Z^{LB} \]

\[ \frac{Z^H}{Z^*} \leq \frac{Z^H}{Z^{LB}} \leq 2 \]