IEOR250 HW5 Suggested Solution

Problem 1

Let $y = (1-\lambda)y_1 + \lambda y_2$. Since g_1 is K_1-convex and g_2 is K_2-convex,

$$
\begin{aligned}
&g_1(y) \leq (1-\lambda)g_1(y_1) + \lambda g_1(y_2) + \lambda K_1 \\
&g_2(y) \leq (1-\lambda)g_2(y_1) + \lambda g_2(y_2) + \lambda K_2 \\
&\alpha g_1(y) + \beta g_2(y) \leq \alpha\{(1-\lambda)g_1(y_1) + \lambda g_1(y_2) + \lambda K_1\} + \beta\{(1-\lambda)g_2(y_1) + \lambda g_2(y_2) + \lambda K_2\} \\
&\leq (1-\lambda)\{\alpha g_1(y_1) + \beta g_2(y_1)\} + \lambda\{\alpha g_1(y_2) + \beta g_2(y_2)\} + \lambda(\alpha K_1 + \beta K_2)
\end{aligned}
$$

thus $\alpha g_1(y) + \beta g_2(y)$ is $(\alpha K_1 + \beta K_2)$-convex

Problem 2

(a) No additional decision variable is needed.

$$
\begin{aligned}
\min \quad & \sum_{i,j} h_i d_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_j y_{ij} = 1 \ \forall\ i \in I \\
& \sum_j X_j = P \\
& \sum_j y_{ij} d_{ij} \leq D_f \ \forall\ i \in I --- \text{additional constraint} \\
& y_{ij} \leq X_j \ \forall \text{ i,j} \\
& X_j \in \{0,1\} \ \forall\ j \in J \\
& y_{ij} \in \{0,1\} \ \forall\ i \in I, j \in J --- \text{or } y_{ij} \geq 0
\end{aligned}
$$

(b) Let M be a sufficiently large number (M$\gg \max_{i,j} d_{ij}$). When we enter the data for d_{ij}s, we can change it to M if $d_{ij} > D_f$.

(c) Let S_i be the state space for demand node i, $S_i=\{j : d_{ij} \leq D_f\}$ $\forall$ i, that is y_{ij} does not exist if $d_{ij} > D_f$, and the problem becomes

$$\begin{aligned}
\min \quad & \sum_{i,j} h_i d_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_{j \in S_i} y_{ij} = 1 \; \forall \; i \in I \\
& \sum_j X_j = P \\
& y_{ij} \leq X_j \; \forall \; i \in I, j \in S_i \\
& X_j \in \{0,1\} \; \forall \; j \in S_i \\
& y_{ij} \in \{0,1\} \; \forall \; i \in I, j \in J --- \text{or } y_{ij} \geq 0
\end{aligned}$$

(d) y_{ij}: 1 location j is a primary facility of demand node i, 0 otherwise.
z_{ij}: 1 location j is a secondary facility of demand node i, 0 otherwise.

$$\begin{aligned}
\min \quad & \sum_{i,j} h_i d_{ij} y_{ij} \\
\text{s.t.} \quad & \sum_j y_{ij} = 1 \; \forall \text{ i} --- \text{each demand node is served by a by primary facility} \\
& \sum_j z_{ij} = 1 \; \forall \text{ i} --- \text{each demand node is served by secondary facility} \\
& \sum_j X_j = P --- \text{build exactly P facilities} \\
& \sum_j y_{ij} d_{ij} \leq D_f \; \forall \text{ i} --- \text{primary facility is at most} D_f \text{distance away} \\
& \sum_j z_{ij} d_{ij} \leq D_s \; \forall \text{ i} --- \text{secondary facility is at most} D_s \text{distance away} \\
& \sum_i y_{ij} \leq F \; \forall \text{ j} --- \text{can only serve as a primary facility for at most F demand nodes} \\
& \sum_i z_{ij} \leq S \; \forall \text{ j} --- \text{can only serve as a primary facility for at most S demand nodes} \\
& y_{ij} + z_{ij} \leq X_j \; \forall \text{ i,j} --- \text{can only serve if the facility is built} \\
& X_j \in \{0,1\} \; \forall \text{ j} \\
& y_{ij}, z_{ij} \in \{0,1\} \; \forall \text{ i,j} --- \text{or } y_{ij}, z_{ij} \geq 0
\end{aligned}$$

Problem 3

(a) Let N be a number that is sufficiently large, $N \gg K$.

x_j: 1 if facility is built at location j, 0 otherwise.
y_{jk}: 1 if a truck from plant k supplies location j, 0 otherwise.

$$
\begin{aligned}
\text{min} \quad & \sum_{j=1}^{J} f_j x_j + \frac{2\alpha}{N} \sum_{j=1}^{J} \sum_{k=1}^{K} d_{jk} y_{jk} \\
\text{s.t.} \quad & \sum_{j} a_{ij} x_j \geq 1 \ \forall \ i = 1, ..., I \\
& \sum_{k} y_{jk} \geq x_j \ \forall \ j = 1, ..., J --- \text{if built, facility j has to be supplied by a truck} \\
& x_j \in \{0, 1\} \ \forall \ j = 1, ..., J \\
& y_{jk} \in \{0, 1\} \ \forall \ j = 1, ..., J, k = 1, ..., K
\end{aligned}
$$

(b) Relax the $\sum_j a_{ij} x_j \geq 1$ ∀i constraints.

$$
\begin{aligned}
\max{}_{u \geq 0} \min{}_{x,y} \quad & \sum_{j} f_j x_j + \frac{2\alpha}{N} \sum_{j,k} d_{jk} y_{jk} + \sum_{i} u_i (1 - \sum_{j} a_{ij} x_j) = \\
\max{}_{u \geq 0} \min{}_{x,y} \quad & \sum_{j} (f_j - \sum_{i} u_i a_{ij}) x_j + \sum_{j,k} (\frac{2\alpha}{N} d_{jk} - u_j) y_{jk} + \sum_{i} u_i \\
\text{s.t.} \quad & \sum_{k} y_{jk} \geq x_j \ \forall \ j = 1, ..., J \\
& x_j \in \{0, 1\} \ \forall \ j = 1, ..., J \\
& y_{jk} \in \{0, 1\} \ \forall \ j = 1, ..., J, k = 1, ..., K
\end{aligned}
$$

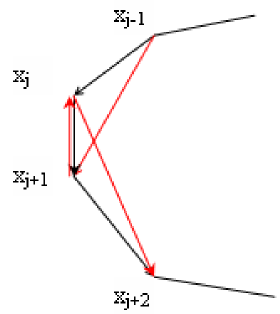
(c) Observe that to solve the subproblem, we would like to set both the x_j's and y_{jk}'s to be one whenever they are multiplying negative coefficients and to be zero otherwise. However, if we set a particular x_j to be one, we force at least one of it's associated y_{jk}'s to be one. This suggests that we should set each x_j to be one only if $(f_j - \sum_i u_i a_{ij})$ is less than zero, and $(f_j - \sum_i u_i a_{ij}) + \min_{j,k}(\frac{2\alpha}{N} d_{jk} - u_j) < 0$.

(d) We've relaxed the constraint that requires all customers to be covered, so some may not be.

Problem 4

Let $x_1, x_2, \ldots, x_n$ be the sequence of cities visited in the optimal tour, and c^* is the length associated with this tour. Let
j=$\text{argmin}_i\{d_{x_i,x_{i+1}} \bigcup d_{x_n,x_1}, i = 1, \ldots, n-1\}$ and
ℓ=$\min\{d_{x_i,x_{i+1}} \bigcup d_{x_n,x_1}, i = 1, \ldots, n-1\}$, then $c^* > n\ell$.

Now let us consider a different tour that visits the cities in the following sequence: $x_1, x_2, ..., x_{j-1}, x_{j+1}, x_j, x_{j+2}, ..., x_n$, and let $\bar{c}$ be the length associated with this tour. $\bar{c} \leq c^{'}$.



The figure above highlights the only difference between the two tours.

$$
\begin{aligned}
\bar{c} - c^* &= (d_{x_{j-1},x_{j+1}} + d_{x_{j+1},x_j} + d_{x_j,x_{j+2}}) - (d_{x_{j-1},x_j} + d_{x_j,x_{j+1}} + d_{x_{j+1},x_{j+2}}) \\
&\leq (d_{x_{j-1},x_j} + d_{x_j,x_{j+1}} + d_{x_j,x_{j+1}} + d_{x_{j+1},x_{j+2}}) - (d_{x_{j-1},x_j} + d_{x_j,x_{j+1}} + d_{x_{j+1},x_{j+2}}) \\
&\leq 2d_{x_j,x_{j+1}} = 2l \\
\frac{\bar{c} - c^*}{c^*} &\leq \frac{2\ell}{c^*} \leq \frac{2}{n}
\end{aligned}
$$