IEOR 250 Assignment3 Suggested Solution

1. (a) Given: \( C=28.5, P=150, V=20, h=0.4 \times 28.5=11.4 \), \( X \sim U(50, 250) \).
   
   Let \( y \) be the number of handbags purchased.

   \[
   F(x) = \Pr(X \leq x) = \frac{x - 50}{200}
   \]

   \[
   E[TC(y)] = Cy + (h - V) \int_0^\infty \max(y - D, 0) f(D) dD - P \int_0^\infty \min(y, D) f(D) dD
   \]

   \[
   \frac{\partial E[TC(y)]}{\partial y} = C + (h - V) \int_0^y f(D) dD - P \int_y^\infty f(D) dD
   \]

   \[
   F(y) = \frac{P - C}{P + h - V} = \frac{y - 50}{200} = \frac{150 - 28.5}{150 + 11.4 - 20} = 0.859 \Rightarrow y = 222
   \]

1. (b) \( X \sim N(150, 20) \), let \( \Phi^{-1}(\cdot) \) be the inverse of the CDF of a standard normal variable.

   \[
   y - 150 \over 20 = \Phi^{-1} \left( y - 150 \over 20 \right) = 1.077 \Rightarrow y = 1.077 \times 20 + 150 = 172
   \]

1. (c) Uniform distribution between 50 and 250 has a larger variance (3333) than normal distribution does (400), thus the buyer has more demand uncertainty.

2. (a) Given: \( r=0.3, p=6, X_{\text{month}} \sim N(28, 8^2) \), \( L=14 \) weeks, \( \pi=10 \), \( K=15 \)

   \[
   X \sim N(28 \cdot 14/4, (8 \cdot \sqrt{14/4})^2) = (98, 15^2)
   \]

   \[
   h = r \cdot p = 1.8, \ D = 28 \cdot 12 = 336
   \]

   \[
   Q_i = \sqrt{\frac{2D}{\pi h}} [K + \pi \cdot b_i], \ Pr(x < R_{i+1}) = 1 - \frac{hQ_i}{\pi D}
   \]

   \[
   z = \Phi^{-1}(Pr(X < R_i)), \ R_{i+1} = z \cdot \sigma + \mu
   \]

   \[
   L(x) = \phi(x) - x[1 - \Phi(x)], \text{ where } \phi(x) \text{ is the PDF of a standard normal random variable, and } \Phi(x) \text{ is CDF of a standard normal variable.}
   \]
\[ R_{i+1} = z \cdot \sigma + \mu, \quad b_{i+1} = \sigma L_i \]

<table>
<thead>
<tr>
<th>i</th>
<th>( b_i )</th>
<th>( Q_i )</th>
<th>( \Pr(x &lt; R_{i+1}) )</th>
<th>( z )</th>
<th>( R_{i+1} )</th>
<th>( L_i )</th>
<th>( b_{i+1} )</th>
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<td>1.749</td>
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<td>1.712</td>
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<td>1.711</td>
<td>123.615</td>
<td>0.0178</td>
<td>0.2662</td>
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\[(Q, R) = (81, 123)\]

2 (b) safety stock: \( s = E(R - X) = R - D_t = 123 - 336 \times \frac{14}{52} = 33 \)

3 (a) \( \alpha = 0.9 \)

\[
\Pr(x \leq R) = \Pr(z \leq \frac{R - \mu}{\sigma}) = 0.9, \text{ where } z \text{ is a standard normal variable.}
\]

\[
z = \frac{R - \mu}{\sigma} = \Phi^{-1}(0.9) \Rightarrow R = \sigma \Phi^{-1}(0.9) + \mu = 134
\]

\[
Q = \sqrt{2KD_h} = 75 \Rightarrow (Q, R) = (75, 134)
\]

3 (b) \( z = \frac{R - \mu}{\sigma} = \frac{134 - 98}{28} = 1.28 \)

\[
\bar{u}(R) = \sigma L(z) = 28 \times (\phi(z) - z[1 - \Phi(z)]) = 1.31
\]

\[
\beta = 1 - \frac{\bar{u}(R)}{Q} = 1 - \frac{1.31}{75} = 98\%
\]

4 Prove by contradiction.

Let \( P \) be the optimal policy, there exists a time \( t \) such that \( y_t I_{t-1}(C_t - y_t) > 0 \) and let \( s \) be the last period before \( t \) that has a positive production level. Let \( a = \min(y_s, C_t - y_t, I_t - 1) > 0 \).

Consider a policy \( P_1 \) that is similar to the current one except for the following changes: decrease \( y_s \) and \( I_k \) \((k = s, s+1, \ldots, t-1)\) by \( a \) and increase \( y_t \) by \( a \) (i.e. shift \( a \) units from time \( s \) to \( t \)). Let \( D_1 \) be difference between the total cost under \( P \) and \( P_1 \).

\[
D_1 = [K \delta(y_s) - K \delta(y_s - a)] + \sum_{k=s}^{t-1}[h_t(I_t) - h_t(I_t - a)] + [K \delta(y_t) - K \delta(y_t + a)]
\]

Note: we only need to consider setup cost and holding cost since demand has to be met, so production cost is the same for both policies.

\[
K \delta(y_s) - K \delta(y_s - a) = \begin{cases} K & \text{if } y_s = a \\ 0 & \text{if } y_s > a \end{cases}
\]

\[
\sum_{k=s}^{t-1}[h_t(I_t) - h_t(I_t - a)] > 0 \text{ since } h_i \text{ is an increasing function for all } i.
\]

and \( K \delta(y_t) - K \delta(y_t + a) = 0 \text{ since } y_t > 0 \). Therefore \( D_1 > 0 \), which contradicts with the assumption that \( P \) is optimal.
5 (a) let \( y_1 \) be the number of units A produced, and let \( y_2 \) be the number of units B produced.

\[
\min(D, y_1 + y_2) = \begin{cases} 
D & \text{if } y_1 + y_2 \geq D \\
y_1 + y_2 & \text{if } y_1 + y_2 < D
\end{cases}
\]

If \( y_1 + y_2 \geq D \), number of units shipped oversea = \( \min(y_2, K) \)

\[
E[TC(y_1, y_2)] = C_A y_1 + C_B y_2 - P_1 \int_{0}^{y_1 + y_2} Df(D)dD - P_1 \int_{y_1 + y_2}^{\infty} (y_1 + y_2)f(D)dD - P_2 \int_{0}^{y_1} \min(y_2, K)f(D)dD - P_2 \int_{y_1}^{y_1 + y_2} \min(y_1 + y_2 - D, K)f(D)dD
\]

5 (b) Claim: it is optimal to produce exactly \( K \) units of product B.

Since we can ship up to \( K \) units oversea, and \( p_2 > C_B \), we are guaranteed to make a profit on those units. It is thus not optimal to produce \( y_2 < K \), thus \( y_2 \geq K \).

Suppose we produce \( K + b \) units of product B. Since we can not ship the extra \( b \) units oversea and we can produce A at a cheaper price \( (C_A < C_B) \) and sell A at a better price \( (p_1 > p_2) \), thus we are better off switching those \( b \) units to A. thus \( y_2 = K \).

5 [c]

\[
E[TC(y_1)] = C_A y_1 + C_B K - P_1 \int_{0}^{y_1 + K} Df(D)dD - P_1 \int_{y_1 + K}^{\infty} (y_1 + K)f(D)dD - P_2 \int_{0}^{y_1} Kf(D)dD - P_2 \int_{y_1}^{y_1 + K} (y_1 + K - D)f(D)dD - P_1 \int_{y_1 + K}^{\infty} Df(D)dD + P_1 \int_{y_1 + K}^{\infty} Df(D)dD
\]

\[
= C_A y_1 + C_B K - P_1 E(D) - P_1 \int_{y_1 + K}^{\infty} (y_1 + K - D)f(D)dD - P_2 \int_{y_1}^{y_1 + K} (y_1 + K - D)f(D)dD
\]

Set \( \frac{\partial E(TC)}{\partial y_1} = 0 \) and solve for \( y_1 \).