IEOR 250 Assignment3 Suggested Solution

1 (a) Given: C=28.5, P=150, V=20, h=0.4×28.5=11.4, $X \sim U(50, 250)$.
Let y be the number of handbags purchased.

$$\begin{aligned}
F(x) &= Pr(X \leq x) = \frac{x-50}{200} \\
E[TC(y)] &= Cy + (h-V)\int_0^\infty \max(y-D,0)f(D)dD - P\int_0^\infty \min(y,D)f(D)dD \\
&= Cy + (h-V)\int_0^y (y-D)f(D)dD - P\int_y^\infty yf(D)dD - P\int_0^y Df(D)dD \\
\frac{\partial E[TC(y)]}{\partial y} &= C + (h-V)\int_0^y f(D)dD - P\int_y^\infty f(D)dD \\
&= C - (h-V)F(y) - P[1-F(y)] = 0 \\
F(y) &= \frac{P-C}{P+h-V} \;\Rightarrow\; \frac{y-50}{200} = \frac{150-28.5}{150+11.4-20} = 0.859 \;\Rightarrow\; y = 222
\end{aligned}$$

1 (b) $X \sim N(150, 20)$, let $\Phi^{-1}(\cdot)$ be the inverse of the CDF of a standard normal variable.

$$\frac{y-150}{20} = \Phi^{-1}(\frac{y-150}{20}) = 1.077 \;\Rightarrow\; y = 1.077 \times 20 + 150 = 172$$

1 (c) Uniform distribution between 50 and 250 has a larger variance (3333) than normal distribution does (400), thus the buyer has more demand uncertainty.

2 (a) Given: r=0.3, p=6, $X_{month} \sim N(28, 8^2)$, L=14 weeks, π=10, K=15

$$X \sim N(28 \cdot 14/4, (8 \cdot \sqrt{\frac{14}{4}})^2) = (98, 15^2)$$

$$h = r \cdot p = 1.8,\ \overline{D} = 28 \cdot 12 = 336$$

$$Q_i = \sqrt{\frac{2\overline{D}}{h}[K + \pi \cdot b_i]},\ Pr(x < R_{i+1}) = 1 - \frac{hQ_i}{\pi\overline{D}}$$

$$z = \Phi^{-1}(Pr(X < R_i)),\ R_{i+1} = z \cdot \sigma + \mu$$

$L(x) = \phi(x) - x[1 - \Phi(x)]$, where $\phi(x)$ is the PDF of a standard normal random variable, and $\Phi(x)$ is CDF of a standard normal variable.

$$R_{i+1} = z \cdot \sigma + \mu, \; b_{i+1} = \sigma L_i$$

i	b_i	Q_i	$\Pr(x < R_{i+1})$	z	R_{i+1}	L_i	b_{i+1}
0	0	74.8	0.9599	1.749	124.186	0.0162	0.2423
1	0.2423	80.7	0.9568	1.715	123.662	0.0176	0.2641
2	0.2641	81.2	0.9565	1.712	123.618	0.0178	0.2660
3	0.2660	81.2	0.9565	1.711	123.618	0.0178	0.2662
4	0.2662	81.2	0.9565	1.711	123.615	0.0178	0.2662

$$(Q, R) = (81, 123)$$

2 (b) safety stock: s $= E(R - X) = R - \overline{D}$t $= 123 - 336 \times \frac{14}{52} = 33$

3 (a) α=0.9

$\Pr(x \leq R) = \Pr(z \leq \frac{R-\mu}{\sigma}) = 0.9$, where z is a standard normal variable.

$$z = \frac{R - \mu}{\sigma} = \Phi^{-1}(0.9) \Rightarrow R = \sigma\Phi^{-1}(0.9) + \mu = 134$$

$$Q = \sqrt{\frac{2KD}{h}} = 75 \Rightarrow (Q, R) = (75, 134)$$

3 (b) z $= \frac{R-\mu}{\sigma} = \frac{134-98}{28} = 1.28$

$$\overline{b}(R) = \sigma L(z) = 28 \times (\phi_(z) - z[1 - \Phi(z)) = 1.31$$

$$\beta = 1 - \frac{\overline{b}(R)}{Q} = 1 - \frac{1.31}{75} = 98\%$$

4 Prove by contradiction.

Let P be the optimal policy, there exists a time t such that $y_t I_{t-1}(C_t - y_t) > 0$ and let s be the last period before t that has a positive production level. Let $a = \min(y_s, C_t - y_t, I_{t-1}) > 0$. Consider a policy P_1 that is similar to the current one except for the following changes: decrease y_s and I_k $(k = s, s+1, ..., t-1)$ by a and increase y_t by a (i.e. shift a units from time s to t). Let D_1 be difference between the total cost under P and P_1.

$$D_1 = [K\delta(y_s) - K\delta(y_s - a)] + \Sigma_{k=s}^{t-1}[h_l(I_l) - h_l(I_l - a)] + [K\delta(y_t) - K\delta(y_t + a)]$$

Note: we only need to consider setup cost and holding cost since demand has to be met, so production cost is the same for both policies.

$$K\delta(y_s) - K\delta(y_s - a) = \begin{cases} K & \text{if } y_s = a \\ 0 & \text{if } y_s > a \end{cases}$$

$\Sigma_{k=s}^{t-1}[h_l(I_l) - h_l(I_l - a)] > 0$ since h$_i$ is an increasing function for all i.

and $K\delta(y_t) - K\delta(y_t + a) = 0$ since y_t >0. Therefore $D_1 > 0$, which contradicts with the assumption that P is optimal.

5 (a) let y_1 be the number of units A produced, and let y_2 be the number of units B produced.

$$\min(D, y_1 + y_2) = \begin{cases} D & \text{if } y_1 + y_2 \geq D \\ y_1 + y_2 & \text{if } y_1 + y_2 < D \end{cases}$$

If $y_1 + y_2 \geq D$, number of units shipped oversea $= \min(y_2, K)$

$$\begin{aligned} E[TC(y_1, y_2)] &= C_A y_1 + C_B y_2 - P_1 \int_0^{y_1+y_2} Df(D)dD - P_1 \int_{y_1+y_2}^{\infty} (y_1 + y_2) f(D)dD \\ &\quad -P_2 \int_0^{y_1} \min(y_2, K) f(D)dD - P_2 \int_{y_1}^{y_1+y_2} \min(y_1 + y_2 - D, K) f(D)dD \end{aligned}$$

5 (b) Claim: it is optimal to produce exactly K units of product B.

Since we can ship up to K units oversea, and $p_2 > C_B$, we are guaranteed to make a profit on those units. It is thus not optimal to produce $y_2 < K$, thus $y_2 \geq K$.
Suppose we produce K+b units of product B. Since we can not ship the extra b units oversea and we can produce A at a cheaper price ($C_A < C_B$) and sell A at a better price ($p_1 > p_2$), thus we are better off switching those b units to A. thus $y_2 = K$.

5 [c]

$$\begin{aligned} E[TC(y_1)] &= C_A y_1 + C_B K - P_1 \int_0^{y_1+K} Df(D)dD - P_1 \int_{y_1+K}^{\infty} (y_1 + K) f(D)dD \\ &\quad -P_2 \int_0^{y_1} K f(D)dD - P_2 \int_{y_1}^{y_1+K} (y_1 + K - D) f(D)dD \\ &\quad -P_1 \int_{y_1+K}^{\infty} Df(D)dD + P_1 \int_{y_1+K}^{\infty} Df(D)dD \\ &= C_A y_1 + C_B K - P_1 E(D) - P_1 \int_{y_1+K}^{\infty} (y_1 + K - D) f(D)dD \\ &\quad -P_2 \int_{y_1}^{y_1+K} (y_1 + K - D) f(D)dD \end{aligned}$$

Set $\frac{\partial E(TC)}{\partial y_1} = 0$ and solve for y_1.