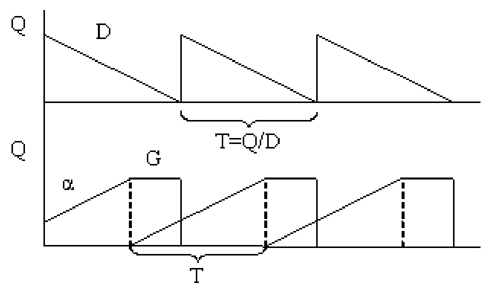
IEOR 250 Assignment2 Suggested Solution

1 We claim that in the optimal strategy α =D.

If α <D then we can not meet the demand. if α >D, we are not optimizing the leasing cost (C$_\alpha$ per unit time), thus α =D.



$$\begin{aligned}
\frac{Cost}{Cycle} &= C_c + \alpha C_\alpha T + \frac{1}{2}hQT = C_c + DC_D T + \frac{1}{2}hQT \\
\min \quad & TC(Q) = \frac{C_c D}{Q} + DC_D + \frac{1}{2}hQ \\
\frac{\partial TC(Q)}{\partial Q} &= \frac{-C_c D}{Q^2} + \frac{1}{2}h \Rightarrow Q^* = \sqrt{\frac{2C_c D}{h}}
\end{aligned}$$

Production Strategy: If $Q^* > C$, start cooking whenever there are C sausages extruded. If $DG \leq Q^* \leq C$, start cooking whenever there are $\sqrt{\frac{2C_c D}{h}}$ sausages extruded. If $Q^* < DG$, start cooking whenever there are DG sausages extruded.

2 Proof.

Let Z* be the optimal solution that minimizes the setup cost, annual inventory holding costs and the annual rent cost, and let B be the rent space associated, i.e. $Z^* = \min \sum_i (\frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2}) + \lambda B$.

In the reader, we saw that a lower bound on the ordering and holding cost is: $sum_i \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2}$ and that a lower bound on the space used is $\frac{1}{2}\sum_i \alpha_i Q_i$.

Thus, we can combine the two to get a lower bound on the cost

$$\begin{aligned}
f(\lambda) &= \sum_i \left(\frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2}\right) + \lambda \sum_i \left(\frac{1}{2}\alpha_i Q_i\right) \\
\frac{\partial f}{\partial Q_i} &= -\frac{K_i D_i}{Q_i^2} + \frac{h_i}{2} + \frac{1}{2}\lambda\alpha_i = 0 \ \forall \ i \Rightarrow Q_i = \sqrt{\frac{2K_i D_i}{h_i + \lambda\alpha_i}} \\
Z^{LB} &= \sum_i \sqrt{2K_i D_i (h_i + \lambda\alpha_i)}
\end{aligned}$$

where the last equality follows from substituting $h_i + \lambda\alpha_i$ for the holding cost in the original EOQ formulation.

Similarly, we can calculate the feasible objective value:

$$\begin{aligned}
Z^{UB} &= \texttt{min} \ \sum_i \left(\frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2} + \lambda\alpha_i Q_i\right) \\
\frac{\partial Z^{UB}}{\partial Q_i} &= -\frac{K_i D_i}{Q_i^2} + \frac{h_i}{2} + \lambda\alpha_i = 0 \ \forall \ i \Rightarrow Q_i = \sqrt{\frac{2K_i D_i}{h_i + 2\lambda\alpha_i}} \\
Z^{UB} &= \sum_i \sqrt{2K_i D_i (h_i + 2\lambda\alpha_i)} \\
\frac{Z^{UB}}{Z^*} &\leq \frac{Z^{UB}}{Z^{LB}} = \frac{\sum_i \sqrt{2K_i D_i (h_i + 2\lambda\alpha_i)}}{\sum_i \sqrt{2K_i D_i (h_i + \lambda\alpha_i)}} \\
&\leq \frac{\sum_i \sqrt{2K_i D_i (2h_i + 2\lambda\alpha_i)}}{\sum_i \sqrt{2K_i D_i (h_i + \lambda\alpha_i)}} = \sqrt{2}
\end{aligned}$$

3 Example.

	P_i	D_i	K_i	h_i	T_{P_i}	T_{D_i}	T_i	$I_i = T\frac{D_i}{P_i}(P_i - D_i)$	$TC_i = \frac{K_i}{T_i} + \frac{h_i I_i}{2}$
Item 1	5	1	60	1	2	10	12	9.6	9.8
Item 2	2	1	3	1	1	2	3	1.5	1.75

$TC = \sum_i TC_i = 11.55$. Note: T_{P_i}'s and T_{D_i}'s are feasible solutions but not necessarily optimal. With Common Cycle Approach, we have

$$T^* = \sqrt{\frac{2\sum_i K_i}{h\sum_i \frac{D_i(P_i - D_i)}{P_i}}} = 9.85$$

$$TC(T^*) = \frac{2\sum_i K_i}{T^*} + \frac{h}{2}\sum_i \frac{D_i(P_i - D_i)}{P_i} = 11.55 \Rightarrow \frac{T^* - TC}{TC} = 11\% > 10\%$$

4 Prove by contradiction.

Let P be the optimal policy, there exists a time t such that $y_t I_{t-1} > 0$ and let s be the last period before t that has a positive production level. Let $a = \min(y_s, y_t, I_{t-1}) > 0$.

Consider the following two policies that are similar to the current one except for the following changes:
P_1: decrease y_s and I_k $(k = s, s+1, ..., t-1)$ by a and increase y_t by a.
P_1: increase y_s and I_k $(k = s, s+1, ..., t-1)$ by a and decrease y_t by a.

Let D_1 be difference between the total cost under P and P_1, and D_2 be the difference between the total cost under P and P_2.
Let $f_k(\cdot)$ be the ordering cost at period k, and $h_k(\cdot)$ be the holding cost at period k.

$$\begin{aligned}
D_1 &= [f_s(y_s) + \Sigma_{k=s}^{t-1} h_k(I_k) + f_t(y_t)] - [f_s(y_s - a) + \Sigma_{k=s}^{t-1} h_k(I_k - a) + f_t(y_t + a)] \\
&= [f_s(y_s) - f_s(y_s - a)] + \Sigma_{k=s}^{t-1}[h_k(I_k) - h_k(I_k - a)] + [f_t(y_t) - f_t(y_t + a)] \\
D_2 &= [f_s(y_s) + \Sigma_{k=s}^{t-1} h_k(I_k) + f_t(y_t)] - [f_s(y_s + a) + \Sigma_{k=s}^{t-1} h_k(I_k + a) + f_t(y_t - a)] \\
&= [f_s(y_s) - f_s(y_s + a)] + \Sigma_{k=s}^{t-1}[h_k(I_k) - h_k(I_k + a)] + [f_t(y_t) - f_t(y_t - a)]
\end{aligned}$$

$$\begin{aligned}
D_1 + D_2 &= [2f_s(y_s) - f_s(y_s - a) - f_s(y_s + a)] + \Sigma_{k=s}^{t-1}[2h(I_k) - h(I_k - a) - h(I_k + a)] \\
&\quad + [2f_t(y_t) - f_t(y_t + a) - f_t(y_t - a)]
\end{aligned}$$

Since both f and h are concave functions:
$2f_s(y_s) - f_s(y_s - a) - f_s(y_s + a) > 0$, $2f_t(y_t) - f_t(y_t + a) - f_t(y_t - a) > 0$ and $2h_k(I_k) - h(I_k - a) - h_k(I_k + a) > 0$ for each k. Thus $D_1 + D_2 > 0$, which means either $D_1 > 0$ or $D_2 > 0$. This contradicts with the assumption that P is optimal.

5 Example.

$$c = \$1/\text{item}, K = \$50, h = \$2/\text{unit time}$$

Period	1	2	3	4	5
D	10	27	11	9	23
Q^{WW}	10	47	0	0	23
Q^{PPB}	37	0	20	0	23

$$TC^{WW} = 288 < TC^{PPB} = 298$$