1 We claim that in the optimal strategy $\alpha = D$.

If $\alpha < D$ then we cannot meet the demand. If $\alpha > D$, we are not optimizing the leasing cost ($C_\alpha$ per unit time), thus $\alpha = D$.

![Diagram of production strategy]

\[
\frac{\text{Cost}}{\text{Cycle}} = C_c + \alpha C_\alpha T + \frac{1}{2} h Q T = C_c + D C_D T + \frac{1}{2} h Q T
\]

\[
\min \quad TC(Q) = \frac{C_c D}{Q} + D C_D + \frac{1}{2} h Q
\]

\[
\frac{\partial TC(Q)}{\partial Q} = -\frac{C_c D}{Q^2} + \frac{1}{2} h \Rightarrow Q^* = \sqrt{\frac{2C_c D}{h}}
\]

**Production Strategy:** If $Q^* > C$, start cooking whenever there are $C$ sausages extruded. If $DG \leq Q^* \leq C$, start cooking whenever there are $\sqrt{\frac{2C_c D}{h}}$ sausages extruded. If $Q^* < DG$, start cooking whenever there are $DG$ sausages extruded.

2 Proof.

Let $Z^*$ be the optimal solution that minimizes the setup cost, annual inventory holding costs and the annual rent cost, and let $B$ be the rent space associated, i.e. $Z^* = \min \sum_i \left( \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2} \right) + \lambda B$. 

1
In the reader, we saw that a lower bound on the ordering and holding cost is: $\sum_i K_i D_i \frac{h_i Q_i}{Q_i} + \frac{h_i Q_i}{2}$ and that a lower bound on the space used is $\frac{1}{2} \sum_i \alpha_i Q_i$.

Thus, we can combine the two to get a lower bound on the cost

$$f(\lambda) = \sum_i \left( \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2} \right) + \lambda \sum_i \left( \frac{1}{2} \alpha_i Q_i \right)$$

$$\frac{\partial f}{\partial Q_i} = -\frac{K_i D_i}{Q_i^2} + \frac{h_i}{2} + \frac{1}{2} \lambda \alpha_i = 0 \quad \forall \ i \Rightarrow Q_i = \sqrt{\frac{2K_i D_i}{h_i + \lambda \alpha_i}}$$

$$Z^{LB} = \sum_i \sqrt{2K_i D_i (h_i + \lambda \alpha_i)}$$

where the last equality follows from substituting $h_i + \lambda \alpha_i$ for the holding cost in the original EOQ formulation.

Similarly, we can calculate the feasible objective value:

$$Z^{UB} = \min \sum_i \left( \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2} + \lambda \alpha_i Q_i \right)$$

$$\frac{\partial Z^{UB}}{\partial Q_i} = -\frac{K_i D_i}{Q_i^2} + \frac{h_i}{2} + \lambda \alpha_i = 0 \quad \forall \ i \Rightarrow Q_i = \sqrt{\frac{2K_i D_i}{h_i + 2\lambda \alpha_i}}$$

$$Z^{UB} = \sum_i \sqrt{2K_i D_i (h_i + 2\lambda \alpha_i)}$$

$$\frac{Z^{UB}}{Z^*} \leq \frac{Z^{UB}}{Z^{LB}} = \frac{\sum_i \sqrt{2K_i D_i (h_i + 2\lambda \alpha_i)}}{\sum_i \sqrt{2K_i D_i (h_i + \lambda \alpha_i)}} \leq \frac{\sum_i \sqrt{2K_i D_i (2h_i + 2\lambda \alpha_i)}}{\sum_i \sqrt{2K_i D_i (h_i + \lambda \alpha_i)}} = \sqrt{2}$$
3 Example.

<table>
<thead>
<tr>
<th>(P_i)</th>
<th>(D_i)</th>
<th>(K_i)</th>
<th>(h_i)</th>
<th>(T_{P_i})</th>
<th>(T_{D_i})</th>
<th>(I_i = T \frac{D_i}{P_i}(P_i - D_i))</th>
<th>(TC_i = \frac{P_i}{T_i} + \frac{h_i I_i}{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Item 1</td>
<td>5</td>
<td>1</td>
<td>60</td>
<td>1</td>
<td>2</td>
<td>10</td>
<td>9.6</td>
</tr>
<tr>
<td>Item 2</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1.5</td>
</tr>
</tbody>
</table>

\(TC = \sum_i TC_i = 11.55\). Note: \(T_{P_i}\)’s and \(T_{D_i}\)’s are feasible solutions but not necessarily optimal. With Common Cycle Approach, we have

\[
T^* = \sqrt{\frac{2 \sum_i K_i}{h \sum_i \frac{D_i(P_i - D_i)}{P_i}}} = 9.85
\]

\[
TC(T^*) = \frac{2 \sum_i K_i}{T^*} + \frac{h}{2} \sum_i \frac{D_i(P_i - D_i)}{P_i} = 11.55 \Rightarrow \frac{T^* - TC}{TC} = 11% > 10%
\]

4 Prove by contradiction.

Let \(P\) be the optimal policy, there exists a time \(t\) such that \(y_t I_{t-1} > 0\) and let \(s\) be the last period before \(t\) that has a positive production level. Let \(a = \min(y_s, y_t, I_{t-1}) > 0\).

Consider the following two policies that are similar to the current one except for the following changes:

- \(P_1\): decrease \(y_s\) and \(I_k\) \((k = s, s+1, \ldots, t-1)\) by \(a\) and increase \(y_t\) by \(a\).
- \(P_2\): increase \(y_s\) and \(I_k\) \((k = s, s+1, \ldots, t-1)\) by \(a\) and decrease \(y_t\) by \(a\).

Let \(D_1\) be difference between the total cost under \(P\) and \(P_1\), and \(D_2\) be the difference between the total cost under \(P\) and \(P_2\).
Let \(f_k(\cdot)\) be the ordering cost at period \(k\), and \(h_k(\cdot)\) be the holding cost at period \(k\).

\[
D_1 = [f_s(y_s) + \Sigma_{k=s}^{t-1} h_k(I_k) + f_t(y_t)] - [f_s(y_s - a) + \Sigma_{k=s}^{t-1} h_k(I_k - a) + f_t(y_t + a)]
\]
\[
= [f_s(y_s) - f_s(y_s - a)] + \Sigma_{k=s}^{t-1} [h_k(I_k) - h_k(I_k - a)] + [f_t(y_t) - f_t(y_t + a)]
\]
\[
D_2 = [f_s(y_s) + \Sigma_{k=s}^{t-1} h_k(I_k) + f_t(y_t)] - [f_s(y_s + a) + \Sigma_{k=s}^{t-1} h_k(I_k + a) + f_t(y_t - a)]
\]
\[
= [f_s(y_s) - f_s(y_s + a)] + \Sigma_{k=s}^{t-1} [h_k(I_k) - h_k(I_k + a)] + [f_t(y_t) - f_t(y_t - a)]
\]

\[
D_1 + D_2 = [2f_s(y_s) - f_s(y_s - a) - f_s(y_s + a)] + \Sigma_{k=s}^{t-1} [2h(I_k) - h(I_k - a) - h(I_k + a)]
\]
\[
+ [2f_t(y_t) - f_t(y_t + a) - f_t(y_t - a)]
\]

Since both \(f\) and \(h\) are concave functions:

\(2f_s(y_s) - f_s(y_s - a) - f_s(y_s + a) > 0, 2f_t(y_t) - f_t(y_t + a) - f_t(y_t - a) > 0\)

and \(2h(I_k) - h(I_k - a) - h(I_k + a) > 0\) for each \(k\). Thus \(D_1 + D_2 > 0\), which means either \(D_1 > 0\) or \(D_2 > 0\). This contradicts with the assumption that \(P\) is optimal.
5 Example.

\[ c = \$1/\text{item}, \; K = \$50, \; h = \$2/\text{unit time} \]

<table>
<thead>
<tr>
<th>Period</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>10</td>
<td>27</td>
<td>11</td>
<td>9</td>
<td>23</td>
</tr>
<tr>
<td>(Q_{WW})</td>
<td>10</td>
<td>47</td>
<td>0</td>
<td>0</td>
<td>23</td>
</tr>
<tr>
<td>(Q_{PPB})</td>
<td>37</td>
<td>0</td>
<td>20</td>
<td>0</td>
<td>23</td>
</tr>
</tbody>
</table>

\[ TC_{WW} = 288 < TC_{PPB} = 298 \]