IEOR 250 Assignment1 Solution

1 (a) Given: B=1000, K=90, r=0.18, $\alpha = 0.6$

	C	h	α	D
tomatoes	0.29	0.29*0.18=0.052	0.6	800
lettuce	0.45	0.45*0.18=0.081	0.6*0.45/0.29=0.931	1350
zucchini	0.25	0.25*0.18=0.045	0.6*0.25/0.29=0.517	670

$$\begin{aligned} min \quad & \sum_{i=1}^{3} \frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2} + C_i D_i \\ s.t. \quad & \sum_{i=1}^{3} \alpha_i Q_i \leq B \\ & Q_i \geq 0 \end{aligned}$$

$$\begin{aligned} f(\lambda) &= \sum_{i=1}^{3} (\frac{K_i D_i}{Q_i} + \frac{h_i Q_i}{2} + C_i D_i) + \lambda(\sum_{i=1}^{3} \alpha_i Q_i - B) \\ \frac{\partial f(\lambda)}{\partial Q_i} &= -\frac{K_i D_i}{Q_i^2} + \frac{h_i}{2} + \lambda \alpha_i = 0 \Rightarrow Q_i = \sqrt{\frac{2KD_i}{h_i + 2\lambda\alpha_i}} \\ \frac{\partial f(\lambda)}{\partial \lambda} &= \sum_{i=1}^{3} \alpha_i Q_i - B = 0 \Rightarrow \sum_{i=1}^{3} \alpha_i Q_i = B \Rightarrow \lambda = 0.476 \\ Q_1 &= \sqrt{\frac{2K_1 D_1}{h_1 + 2\lambda\alpha_1}} = \sqrt{\frac{2(90)(800)}{0.052 + 2(0.476)(0.5)}} = 480.60 \\ Q_2 &= \sqrt{\frac{2K_2 D_2}{h_2 + 2\lambda\alpha_2}} = \sqrt{\frac{2(90)(1350)}{0.081 + 2(0.476)(0.776)}} = 501.19 \\ Q_3 &= \sqrt{\frac{2K_3 D_3}{h_3 + 2\lambda\alpha_3}} = \sqrt{\frac{2(90)(670)}{0.045 + 2(0.476)(0.431)}} = 473.70 \\ TC &= \sum_{i=1}^{3} TC_i(Q_i) = \sum_{i=1}^{3} (\frac{KD_i}{Q_i} + \frac{h_i Q_i}{2} + C_i D_i) = 394.3 + 870.22 + 305.45 = 1570.03 \end{aligned}$$

1 (b) Try Equal Order Interval Method

$$\begin{aligned} t &= \frac{2B \cdot \sum_{i=1}^{3} \alpha_i Di}{(\sum_{i=1}^{3} \alpha_i Di)^2 + \sum_{i=1}^{3} \alpha_i^2 Di^2} \\ TC &= \sum_{i=1}^{3} TC_i(t) = \sum_{i=1}^{3} (\frac{K}{t} + \frac{h_i D_i t}{2} + C_i D_i) = 381.32 + 779.28 + 312.96 = 1473.56 \end{aligned}$$

This shows that we can achieve a lower cost solution using the Equal Order Interval method.

2

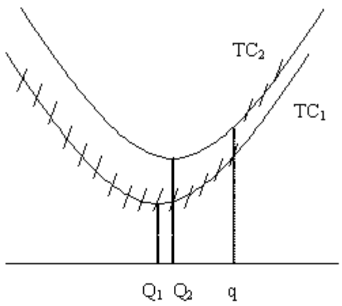
$$TC(Q) = \frac{KD}{Q} + \frac{hQ}{2} + CD$$
$$TC'(Q) = -\frac{KD}{Q^2} + \frac{h}{2} = 0 \Rightarrow Q^* = \sqrt{\frac{2KD}{h}}$$
$$\texttt{holding cost per unit time} = \frac{hQ}{2} = \frac{h}{2}\sqrt{\frac{2KD}{h}} = \sqrt{2KDh}$$
$$\texttt{fixed cost per unit time} = \frac{KD}{Q} = KD\sqrt{\frac{h}{2KD}} = \sqrt{2KDh}$$

3 (a) All units "reverse discount"

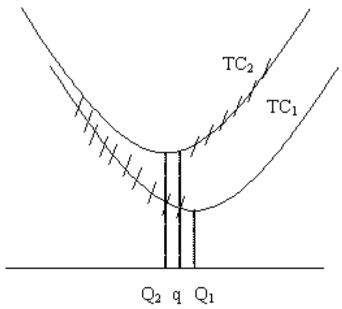
$$\begin{aligned} Cost &= K + C_1Q \texttt{ if } Q < q \\ &= K + C_1Q \texttt{ if } Q \geq q \\ TC_1(Q) &= \frac{KD}{Q} + \frac{h_1Q}{2} + C_1D \Rightarrow Q_1^c = \sqrt{\frac{2KD}{h_1}} \\ TC_2(Q) &= \frac{KD}{Q} + \frac{h_2Q}{2} + C_2D \Rightarrow Q_2^c = \sqrt{\frac{2KD}{h_2}} \end{aligned}$$

3 (b)

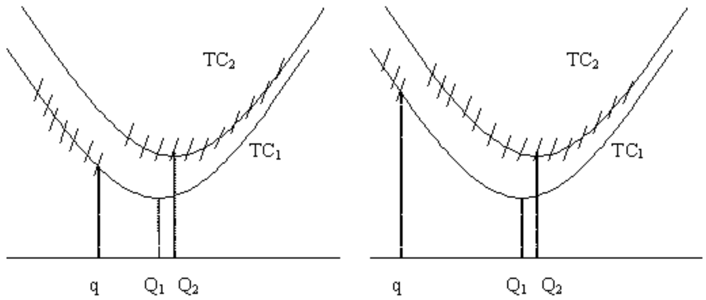
- If $Q_1^c < q$, order Q_1^c.



- If $Q_1^c \geq q$ and $Q_2^c < q$, order q.



- If $Q_1^c \geq q$ and $Q_2^c \geq q$, order Q_2^c if $TC_2(Q_2^c) < TC_1(q)$; else order q.



3 (c)

$$\begin{aligned}
&K = 10, D = 50000, C_1 = 1, C_2 = 1.005, h = 0.2, q = 1500 \\
&Q_1^c = Q_2^c = \sqrt{\frac{2KD}{h}} = 2236 > q = 1500 \\
&TC_2(Q_2^c) = \frac{KD}{Q_2} + \frac{hQ_2}{2} + C_2 D = 50697.21 \\
&TC_1(q) = \frac{KD}{q} + \frac{hq}{2} + C_1 D = 50483.33 \texttt{ thus, order } 1500.
\end{aligned}$$

4 Let $Q' = kQ$ be the new order quantity, $T' = kT$ be the new length of the ordering interval, and C' be the new per unit cost.

$$\begin{aligned}
TC(Q) &= \frac{KD}{Q} + CD + \frac{hQ}{2} \\
TC(Q') &= \frac{K}{T'} + \frac{C'Q'}{T'} + \frac{hQ'}{2} = \frac{K}{kT} + \frac{C'kQ}{kT} + \frac{hkQ}{2} = \frac{KD}{kQ} + C'D + \frac{hkQ}{2} \\
&= \frac{KD}{k}\sqrt{\frac{h}{2KD}} + C'D + \frac{hk}{2}\sqrt{\frac{2KD}{h}} = (k + \frac{1}{k})\sqrt{\frac{hKD}{2}} + C'D \\
&= TC(Q^*) = \sqrt{2KDh} + CD \\
\Rightarrow C' &= C - \frac{(k-1)^2}{k}\sqrt{\frac{hK}{2D}} \Rightarrow \frac{C' - C}{C} = 1 - \frac{\frac{(k-1)^2}{k}\sqrt{\frac{hK}{2D}}}{C}
\end{aligned}$$

Fractional Discount $= \frac{(k-1)^2}{kC}\sqrt{\frac{hK}{2D}}$

5 (a) Ignore the constant ordering cost CD.

$$\frac{TC'(Q')}{TC(Q)} = \frac{\frac{KD}{Q'} + CD + \frac{hQ'}{2}}{\frac{KD}{Q} + CD + \frac{hQ}{2}} \approx \frac{\frac{KD}{Q'} + \frac{hQ'}{2}}{\frac{KD}{Q} + \frac{hQ}{2}} = \frac{\sqrt{2K'Dh}}{\sqrt{2KDh}} = \sqrt{\frac{K'}{K}}$$

5 (b) The derivative of function $y = \sqrt{x}$ is $y' = \frac{1}{2}x^{-\frac{1}{2}}$. The value of y' decreases in x around x=1, it is better to overestimate K than underestimate.

6

$$\begin{aligned}
TC(K,Q) &= \frac{KD}{Q} + \frac{hQ}{2} + CD + A - B\ln K \\
&= \sqrt{2KDh} + CD + A - B\ln K \\
\frac{\partial TC(K,Q)}{\partial Q} &= -\frac{KD}{Q^2} + \frac{h}{2} = 0 \Rightarrow Q^* = \frac{2KD}{h} \\
\frac{\partial TC(K,Q)}{\partial K} &= \sqrt{\frac{Dh}{2K}} - \frac{B}{K} = 0 \Rightarrow K^* = \frac{2B^2}{hD} \Rightarrow Q^* = \frac{2B}{h}
\end{aligned}$$

optimal setup cost $= \frac{2B^2}{hD}$ and optimal order quantity $= \frac{2B}{h}$.

7 (a) Suppose $Q^* = mq$ is the optimal order, then $TC(mq) \le TC((m-1)q)$ and $TC(mq) \le TC((m+1)q)$. Let $f(Q) = TC(Q)$

$$\begin{aligned}
f(mq) \le f((m+1)q) &\Rightarrow \frac{KD}{mq} + \frac{hmq}{2} \le \frac{KD}{(m+1)q} + \frac{h(m+1)q}{2} \\
\Rightarrow \frac{KD}{q}[\frac{1}{m} - \frac{1}{m+1}] \le \frac{hq}{2} &\Rightarrow q^2 \ge \frac{2KD}{h}\frac{1}{m(m+1)} \\
\Rightarrow q \ge \sqrt{\frac{2KD}{h}}\sqrt{\frac{1}{m(m+1)}} &\Rightarrow mq \ge \sqrt{\frac{2KD}{h}}\sqrt{\frac{m}{m+1}} \\
\Rightarrow \frac{Q^*}{Q^e} \ge \sqrt{\frac{m}{m+1}} &\Rightarrow \frac{Q^e}{Q^*} \le \sqrt{\frac{m+1}{m}} \qquad (1) \\
f(mq) \le f((m-1)q) &\Rightarrow \frac{KD}{mq} + \frac{hmq}{2} \le \frac{KD}{(m-1)q} + \frac{h(m-1)q}{2} \\
\Rightarrow \frac{KD}{q}[\frac{1}{m-1} - \frac{1}{m}] \ge \frac{hq}{2} &\Rightarrow q^2 \le \frac{2KD}{h}\frac{1}{m(m-1)} \\
\Rightarrow q \le \sqrt{\frac{2KD}{h}}\sqrt{\frac{1}{m(m-1)}} &\Rightarrow mq \le \sqrt{\frac{2KD}{h}}\sqrt{\frac{m}{m-1}} \\
\Rightarrow \frac{Q^*}{Q^e} \le \sqrt{\frac{m}{m-1}} &\Rightarrow \frac{Q^e}{Q^*} \ge \sqrt{\frac{m-1}{m}} \qquad (2)
\end{aligned}$$

$$(1) + (2) \Rightarrow \sqrt{\frac{m-1}{m}} \le \frac{Q^e}{Q^*} \le \sqrt{\frac{m+1}{m}}$$

7 (b) Since $Q^e\sqrt{\frac{m}{m+1}} \le Q^* \le Q^e\sqrt{\frac{m}{m-1}}$, either $TC(Q^*) \le TC(Q^e\sqrt{\frac{m}{m-1}})$ or $TC(Q^*) \le TC(Q^e\sqrt{\frac{m}{m+1}})$

- $TC(Q^*) \le TC(Q^e\sqrt{\frac{m}{m-1}}) \Rightarrow$

$$\begin{aligned}
TC(Q^*) &\le \frac{KD}{Q^e\sqrt{\frac{m}{m-1}}} + \frac{hQ^e\sqrt{\frac{m}{m-1}}}{2} = \frac{KD}{\sqrt{\frac{2KD}{h}}\sqrt{\frac{m}{m-1}}} + \frac{h\sqrt{\frac{2KD}{h}}\sqrt{\frac{m}{m-1}}}{2} \\
&= \sqrt{\frac{2KD}{h}}[\sqrt{\frac{m}{m-1}} + \sqrt{\frac{m-1}{m}}] = \frac{1}{2}\sqrt{2KDh}[\sqrt{\frac{m}{m-1}} + \sqrt{\frac{m-1}{m}}]
\end{aligned}$$

This is an increasing function in m, given m≥2.
Thus $TC(Q^*) \leq \frac{TC(Q^e)}{2}(\frac{1}{\sqrt{2}} + \sqrt{2}) = 1.06TC(Q^e)$

- $TC(Q^*) \leq TC(Q^e\sqrt{\frac{m}{m+1}}) \Rightarrow$

$$
\begin{aligned}
TC(Q^*) &\leq \frac{KD}{Q^e\sqrt{\frac{m}{m+1}}} + \frac{hQ^e\sqrt{\frac{m}{m+1}}}{2} = \frac{KD}{\sqrt{\frac{2KD}{h}}\sqrt{\frac{m}{m+1}}} + \frac{h\sqrt{\frac{2KD}{h}}\sqrt{\frac{m}{m+1}}}{2} \\
&= \sqrt{\frac{2KD}{h}}[\sqrt{\frac{m}{m+1}} + \sqrt{\frac{m+1}{m}}] = \frac{1}{2}\sqrt{2KDh}[\sqrt{\frac{m}{m+1}} + \sqrt{\frac{m+1}{m}}]
\end{aligned}
$$

This is an increasing function with respect to m.
Thus $TC(Q^*) \leq \frac{TC(Q^e)}{2}(\sqrt{\frac{3}{2}} + \sqrt{\frac{2}{3}}) = 1.02TC(Q^e) \leq 1.06TC(Q^e)$