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Basis Paths and a Polynomial Algorithm for the Multistage Production-Capacitated Lot-Sizing Problem

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We consider the multilevel lot-sizing problem with production capacities (MLSP-PC), in which production and transportation decisions are made for a serial supply chain with capacitated production and concave cost functions. Existing approaches to the multistage version of this problem are limited to nonspeculative cost functions—up to now, no algorithm for the multistage version of this model with general concave cost functions has been developed. In this paper, we develop the first polynomial algorithm for the MLSP-PC with general concave costs at all of the stages, and we introduce a novel approach to overcome the limitations of previous approaches. In contrast to traditional approaches to lot-sizing problems, in which the problem is decomposed by time periods and is analyzed unidirectionally in time, we solve the problem by introducing the concept of a basis path, which is characterized by time and stage. Our dynamic programming algorithm proceeds both forward and backward in time along this basis path, enabling us to solve the problem in polynomial time.

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1. Introduction

In the deterministic multilevel lot-sizing problem with production capacities (MLSP-PC), the optimal manufacturing and distribution plan is determined for a centralized serial supply chain with a capacitated manufacturing stage, several intermediate distribution stages (representing a distribution center, wholesaler, etc.), and a final retail stage. This optimal plan specifies production quantities for the manufacturing stage of the supply chain, and a distribution plan for the entire supply chain to meet time-varying demand that minimizes total cost, including production, transportation, and inventory holding costs.

The single-stage uncapacitated lot-sizing problem was introduced by Wagner and Whitin (1958), and efficient solution algorithms were designed by Federgruen and Tzur (1991), Wagelmans et al. (1992), and Aggarwal and Park (1993). The multistage version of the uncapacitated problem was solved by Zangwill (1968, 1969). Florian and Klein (1971) addressed the capacitated single-stage version of the problem (see also Chung and Lin 1988 and van Hoesel and Wagelmans 1996). Optimal algorithms for the multistage problem with production capacity were first presented by Kaminsky and Simchi-Levi (2003) for the two-stage case (2LSP-PC). van Hoesel et al. (2005) generalized the 2LSP-PC to the multistage lot-sizing problem MLSP-PC and Sargut and Romeijn (2007) extended the 2LSP-PC to allow for subcontracting. For a two-stage lot-sizing model with outbound transportation, Lee et al. (2003) consider cargo capacity constraints.

In this paper, we consider the MLSP-PC with general concave costs. For the version of the problem with an affine transportation cost function, linear inventory costs, and no speculative motive (that is, variable costs and holding costs are such that it is economical to keep inventories upstream as late as possible, which we will refer to as a *nonspeculative transportation cost structure* or simply a *nonspeculative cost structure* for the remainder of the paper), van Hoesel et al. (2005) developed a polynomial time algorithm. For models with general concave production, transportation, and inventory costs, however, no polynomial algorithm has been discovered up to now for problems with more than two stages. Although the nonspeculative cost structure described above can model the value-added flow in supply chains, it does not always effectively model the impact of transportation or holding costs that change dramatically over time, or general economies of scale in transportation. For example, if fuel prices are seasonal, it may make sense to speculatively ship in advance of fuel price increases. In this paper, we develop the first polynomial algorithm for the MLSP-PC with general concave costs at all of the stages, and introduce a novel approach to overcome the limitations of previous approaches in the literature, an approach that has the potential to be more broadly applied.

Most lot-sizing problems are modeled as discrete-time dynamic programs and are solved by iteratively enumerating over time periods. For instance, when solving the single-stage capacitated problem defined in Florian and Klein (1971), one needs to solve the optimality equation for each state (that is, cumulative production quantity) in a given period. Then, the same computations are repeated for each subsequent period to determine the optimal policy and the resulting production schedule. This time-based approach is reflected by the fact that time often appears as the subscript in the notation for the value function. Indeed, van Hoesel et al. (2005) show that a traditional time-based enumeration solves the MLSP-PC with nonspeculative transportation costs in polynomial time, using the fact that this multistage lot-sizing problem with fixed-charge (affine) transportation and linear inventory costs is fully specified by characterizing manufacturing decisions. However, under a general concave cost structure, manufacturing decisions no longer characterize the entire plan. In order to solve the DP for this model, we need to keep track of production and transportation decisions at all stages. Consequently, there is no polynomial algorithm that will solve this problem by performing recursive calculations sequentially iterating over time periods.

In contrast, in this paper we propose a novel approach for conducting iterative computations to solve the MLSP-PC. Instead of iterating over time, we iterate along path in the two-dimensional space of time and stage in the supply chain, which we call a *basis path*. Consequently, *in contrast to every other lot-sizing DP that we are aware of, our algorithm requires us to in general iterate both forward and backward in time*. This approach alone does not directly yield a polynomial algorithm because there are a large (indeed, exponential) number of basis paths. However, we show that this new approach enables us to consider a sufficiently small set of possible basis paths to find the optimal solution of the MLSP-PC with general concave costs, resulting in a polynomial-time algorithm.

In the next section, we formulate the MLSP-PC and characterize some basic properties of the model. In §3, we introduce key basis path concepts, and the notion of partial trees to describe partial production and distribution plans. In §4, we explain how the optimal schedule can be found for a given basis path. In §5, we build on the previous section's results to develop a polynomial-time algorithm for the MLSP-PC. (In Appendix S.3 we present several ways to further reduce the complexity of the algorithm.) We conclude in §6. An electronic companion to this paper is available as supplemental material at http://dx.doi.org/10.1287/opre.1120.1141.

2. Problem Formulation and Solution Structures

2.1. Notation and Problem Formulation

Let T denote the length of the planning horizon, and let L denote the number of stages in a serial supply chain, where manufacturing occurs at stage 1 and external orders are faced at stage L. To clarify the exposition, we use index i only to denote stages from 1 to L; and use indices j, s, and t for time periods from 1 to T. For each stage $i \in \{1, 2, \ldots, L\}$ and period $j \in \{1, 2, \ldots, T\}$ we define the following notation:

- d_j: demand faced by the retailer (stage L) in period j.
- C: production capacity at the first stage.
- x_{ij}: production or transportation quantity at stage i period j. If $i = 1$, this is the production quantity; otherwise, if $i > 1$, this is the transportation quantity to supply chain stage i from stage $i-1$ at time j.
- I_{ij}: the amount of inventory at stage i at the end of period j.
- $p_{ij}(x_{ij})$: concave production or transportation cost function at stage i in period j for the amount $x_{ij} \geqslant 0$. Furthermore, the production and transportation cost functions can include a fixed setup cost. In this case, $p_{ij}(x_{ij}) = K_{ij} + c_{ij}(x_{ij})$ if $x_{ij} > 0$ and $p_{ij}(x_{ij}) = 0$ if $x_{ij} = 0$, where K_{ij} is the setup cost and $c_{ij}(x_{ij})$ is a nonnegative concave function for $x_{ij} \in (0, \infty)$.
- $h_{ij}(I_{ij})$: concave inventory holding cost function at stage i for inventory amount $I_{ij} \geqslant 0$ at the end of period j.

Given an interval $\mathcal{J} = [t_1, t_2]$, $d_{\mathcal{J}}$ denotes the total demand during the interval, i.e., $d_{\mathcal{J}} = d_{t_1} + \cdots + d_{t_2}$. For clarity, we sometimes denote the total sum explicitly by $d_{[t_1, t_2]}$. We assume that the production capacity is stationary and equal to C units per period, and that $d_{[1,j]} \leqslant jC$ for each j to ensure feasibility.

Given these definitions, in the MLSP-PC, we determine a production and distribution plan so that the total cost through the supply chain is minimized:

(MLSP-PC)

$$\min \sum_{i=1}^{L}\sum_{j=1}^{T}[p_{ij}(x_{ij}) + h_{ij}(I_{ij})] \tag{1a}$$

$$\text{s.t.}\quad I_{i,j-1} + x_{ij} = x_{i+1,j} + I_{ij} \qquad i = 1, \ldots, L-1,\; j = 1, \ldots, T, \tag{1b}$$

$$I_{L,j-1} + x_{L,j} = d_j + I_{L,j} \quad j = 1, \ldots, T, \tag{1c}$$

$$x_{1,j} \leqslant C \quad j = 1, \ldots, T, \tag{1d}$$

$$I_{i,0} = I_{i,T} = 0 \quad i = 1, \ldots, L, \tag{1e}$$

$$x_{ij} \geqslant 0, I_{ij} \geqslant 0, \quad i = 1, \ldots, L, j = 1, \ldots, T. \tag{1f}$$

Equations (1b) and (1c) balance inventory over time and supply chain stages, and Equation (1d) constrains production to be no more than capacity. Note that if this capacity constraint is relaxed, our problem reduces to the multistage uncapacitated problem of Zangwill (1969). In a solution $x = (x_{ij})$, a period t is called a *production period* if $x_{1,t} > 0$. A production period t is called a *full production period* if $x_{1,t} = C$; otherwise, if $0 < x_{1,t} < C$, period t is called a *partial production period*.

The MLSP-PC belongs to the class of minimum concave cost flow problems, which are known to be NP-hard (Nemhauser and Wolsey 1988). If every cost function consists of a setup cost and a variable unit cost (so that $c_{ij}(x_{ij}) = c_{ij} \cdot x_{ij}$ where c_{ij} is a constant per unit cost), then the MLSP-PC is said to have a *fixed-charge* cost structure. By convention, for a fixed-charge cost structure, the holding cost is assumed to be linear so that $h_{ij}(I_{ij}) = h_{ij} \cdot I_{ij}$ where h_{ij} is a per-unit storage cost. It has been an open question as to whether or not a polynomial algorithm exists to solve the MLSP-PC with general concave production costs and fixed-charge transportation costs. Until now, a polynomial algorithm has only been found (by van Hoesel et al. 2005) for the special case where the transportation costs have no speculative motive (referred to as *nonspeculative transportation* cost structure or simply *nonspeculative*). Formally, a nonspeculative cost structure is one in which

$$c_{i,j} + h_{i,j} \geqslant h_{i-1,j} + c_{i,j+1}$$

for $i = 2, \ldots, L$ and $j = 1, \ldots, T-1$, suggesting that it is economical to keep inventories in upstream warehouses as late as possible, an observation that as it turns out makes the problem significantly more straightforward to solve, but that fails to model various settings as we discussed above in the introduction.

In this paper, we provide an $O(LT^8)$ algorithm for the MLSP-PC with general concave costs, and thus as special cases the MLSP-PC with the concave production and (general, not necessarily nonspeculative) fixed-charge transportation cost structure and the MLSP-PC with the fixed-charge cost structure at all stages.

2.2. Structure of Extreme Points

2.2.1. Minimum Concave Cost Network. The feasible region defined by constraints (1b)–(1f) is a bounded polyhedron, and thus it is compact and convex with finite extreme points. Because the objective function is concave, it is minimized at an extreme point solution (Zangwill 1966). To characterize the properties of the extreme points, we view the MLSP-PC defined in (1a)–(1f) as a minimum concave cost network flow problem (as in van Hoesel et al. 2005). Figure 1 illustrates a network representation of a 10-period problem with 3 stages. The node at stage i in period j is denoted by (i, j), and has an entering arc with production or transportation quantity x_{ij}. Given this network representation of the problem, a production and distribution plan is a distribution of the $d_{[1,10]}$ units from the *manufacturer's nodes* $(1, j)$ via *intermediate nodes* (i, j), $1 < i < L$, and to the *retailer's nodes* (L, j).

Figure 2 illustrates an extreme point solution of the MLSP-PC. In this figure, each gray node, $(1, j)$, represents a period with full production, and each black node represents a period with partial production. For instance, in periods 2, 6, and 8 production is at capacity, whereas in periods 1 and 4 production is less than capacity, but greater than 0. Notice that this network contains several (undirected) cycles; for instance, the flows of $x_{1,2}$, $I_{1,2}$, $x_{2,3}$, $I_{2,3}$, $x_{2,4}$, and $x_{1,4}$ make a cycle.

Figure 1. The network of production and transportation with inventory.

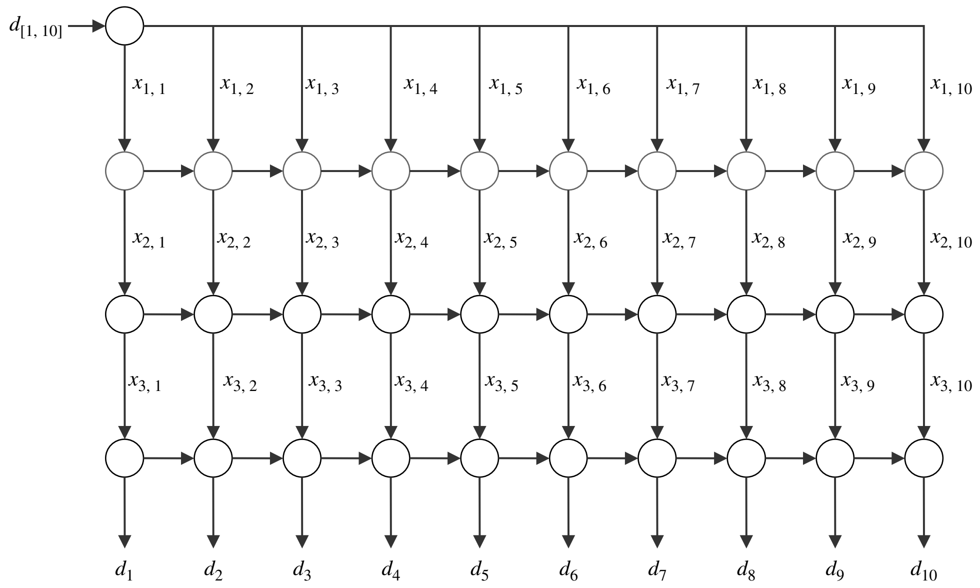
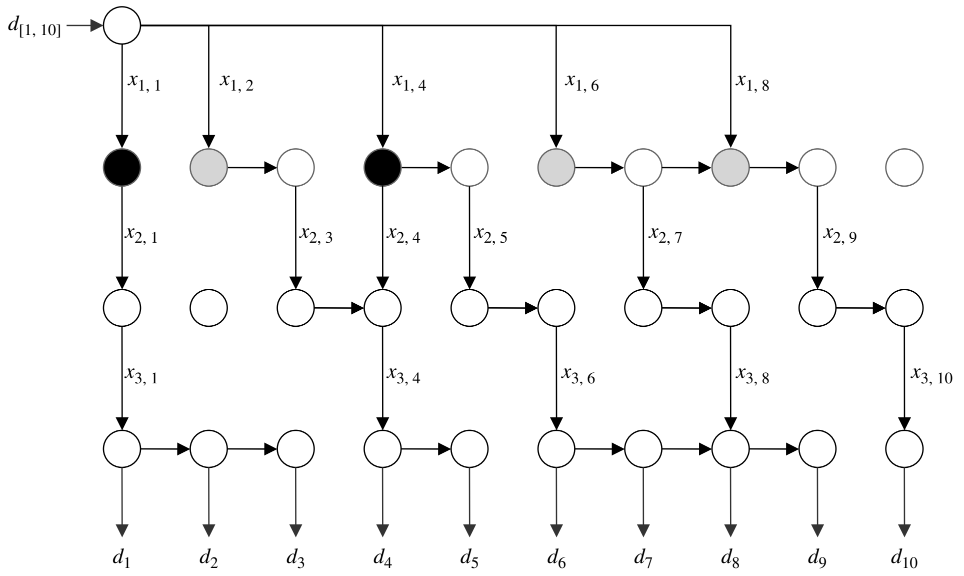


Figure 2. Network flows of a typical extreme point solution.



It can be shown that the subnetwork involving only *free* (or unsaturated) flows of an extreme point solution contains no cycle (Zangwill 1968, Ahuja et al. 1993) where a flow of production, transportation, or inventory is a free flow if it is strictly between its lower and upper bounds. Because transportation and inventory quantities have no upper bound, any one of these quantities greater than 0 yields a free flow. The production quantity, however, has upper bound C; consequently, only partial production yields a free flow. Figure 3 gives the subnetwork of free flows only.

Observe that the subnetwork in Figure 3 has no cycles. In order to focus on critical features of a minimum concave cost network flow problem, we disregard the flows corresponding to productions $x_{1,j}$ in the first stage and the flows corresponding to demands. This results in a *reduced subnetwork* of an extreme point solution (see Figure 4).

2.2.2. Regeneration Network. Each connected component of the reduced subnetwork is called a *regeneration network*, analogous to a *regeneration interval* in single-stage capacitated lot-sizing problems. A production and distribution plan can be described by a set of regeneration networks. Given a plan, we identify regeneration networks by the earliest and latest periods in the manufacturer's and retailer's stages. In particular, $\mathcal{N} = (s_1, s_2, t_1, t_2)$ identifies a regeneration network with the *manufacturer's interval* $[s_1, s_2]$ and the *retailer's interval* $[t_1, t_2]$ where s_1 and s_2 are the earliest and latest periods of the network in the

Figure 3. Network of free flows.

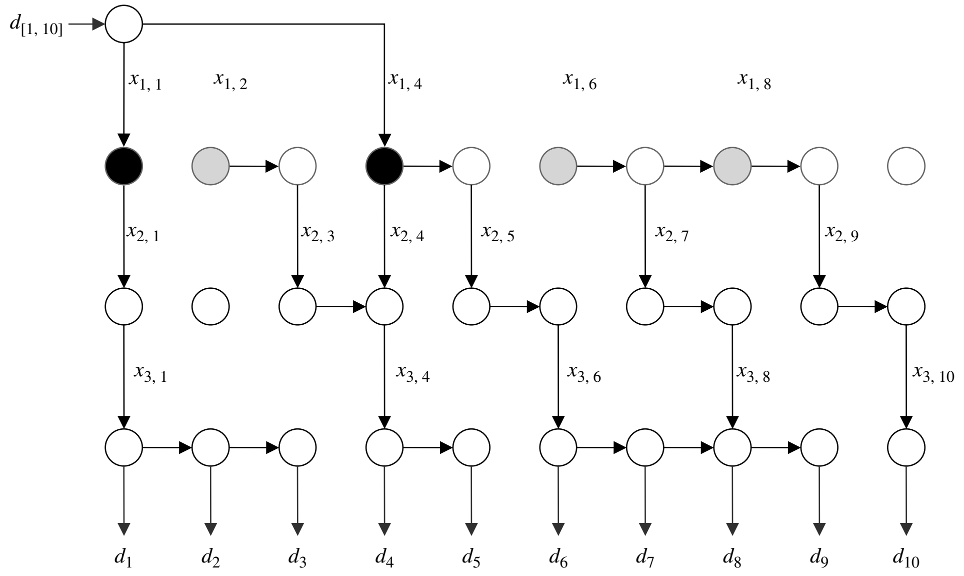
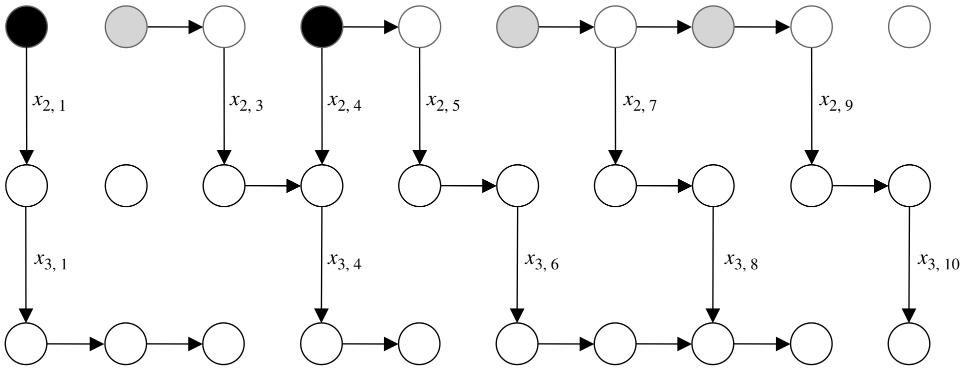


Figure 4. A reduced subnetwork of inventory and transportation flows.



manufacturer's horizon, and t_1 and t_2 are the earliest and latest periods of the network in the retailer's horizon. In Figure 4, there are two regeneration networks $(1, 1, 1, 3)$ and $(2, 9, 4, 10)$. Note that for any regeneration network (s_1, s_2, t_1, t_2), by definition, we have $1 \leqslant s_1 \leqslant s_2 \leqslant T$, $1 \leqslant t_1 \leqslant t_2 \leqslant T$, $s_1 \leqslant t_1$, and $s_2 \leqslant t_2$. Nodes between two consecutive regeneration networks, $\mathcal{N}$ and $\mathcal{N}'$, do not have any flows associated with them. We adopt the convention that any node *between* two regeneration networks belong to the earlier of the two networks. That is, given two networks $\mathcal{N} = (s_1, s_2, t_1, t_2)$ and $\mathcal{N}' = (s_1', s_2', t_1', t_2')$, we extend $\mathcal{N}$ to include the nodes $(1, s_2+1), \ldots, (1, s_1'-1)$ and $(L, t_2+1), \ldots, (L, t_1'-1)$, so that we assume $s_2 = s_1' - 1$ and $t_2 = t_1' - 1$.

As mentioned in §2.2.1 (based on the results of Zangwill 1968 and Ahuja et al. 1993), the network of free flows for the extreme point solution has no cycles. Because a regeneration network of the extreme point solution is a component of the reduced subnetwork from the network of free flows, we can see that the regeneration network also contains no cycle. If the regeneration network has more than one partial production period, then the original network of free flows will contain a cycle, which means that the corresponding solution is not an extreme point solution. Thus, there must be an optimal extreme point solution that has at most one partial production, or more formally:

Proposition 1. *For the MLSP-PC, there exists an* (*extreme point*) *optimal solution such that each regeneration network has no cycle and contains at most one partial production.*

From now on, we consider only the solutions satisfying Proposition 1 and we assume that each regeneration network is derived from an extreme point solution. Consider a regeneration network $\mathcal{N}$ with retailer's interval $\mathcal{J} = [t_1, t_2]$. The assumption of stationary capacity C, together with Proposition 1, reduces the possible choices of production quantity in each period in a given regeneration network: it should be either zero, or the full production quantity C, or the partial production quantity, denoted $\epsilon_{\mathcal{J}}$. Because the network $\mathcal{N}$ has at most one partial production, it follows that the total production quantity should be $kC + \epsilon_{\mathcal{J}}$ for some nonnegative integer k, which is allocated to the total amount $d_{\mathcal{J}}$ of demand. Hence, we have $\epsilon_{\mathcal{J}} = d_{\mathcal{J}} - \lfloor d_{\mathcal{J}}/C \rfloor C$. To sum up, a production quantity of a regeneration network with retailer's interval $\mathcal{J}$ should be one of $\{0, \epsilon_{\mathcal{J}}, C\}$ where $\epsilon_{\mathcal{J}} = d_{\mathcal{J}} - \lfloor d_{\mathcal{J}}/C \rfloor C$.

The DP algorithms in §§4 and 5 use state variables representing cumulative production quantities. In preparation, it is helpful to identify possible cumulative production quantities. Let $\Omega_{\mathcal{J}}$ be the set of all possible cumulative production quantities of a regeneration network with the retailer's interval $\mathcal{J}$:

$$\Omega_{\mathcal{J}} = \{kC\colon 0 \leqslant k \leqslant \lfloor d_{\mathcal{J}}/C \rfloor\} \cup \{\epsilon_{\mathcal{J}} + kC\colon 0 \leqslant k \leqslant \lfloor d_{\mathcal{J}}/C \rfloor\}.$$

Note that the number of elements in the set $|\Omega_{\mathcal{J}}| = O(T)$.

2.3. The Dynamic Programming Algorithm for the General MLSP-PC

Let $F(s, t)$ be the cost of the minimum-cost solution satisfying demands $d_1, d_2, \ldots, d_t$ using production in periods $1, 2, \ldots, s$ in the manufacturer's horizon and let $f(\mathcal{N})$ be the minimum cost of regeneration network $\mathcal{N}$. Then, as in van Hoesel et al. (2005), the following recursion can be used to determine an optimal solution: For $1 \leqslant s_2 \leqslant t_2 \leqslant T$,

$$F(0, 0) = 0, \quad \text{and}$$

$$F(s_2, t_2) = \min_{1 \leqslant s_1 \leqslant s_2,\, 1 \leqslant t_1 \leqslant t_2} \{F(s_1 - 1, t_1 - 1) + f(\mathcal{N})\colon \mathcal{N} = (s_1, s_2, t_1, t_2)\} \quad (2)$$

where the optimal solution is $F(T, T)$. If we could compute the minimum cost $f(\mathcal{N})$ of each regeneration network $\mathcal{N} = (s_1, s_2, t_1, t_2)$, we could solve the MLSP-PC using the usual dynamic programming approach. Of course, this assumes that we are given the cost of each regeneration network $f(\mathcal{N})$. Until this paper, however, no polynomial algorithm for computing this cost has been developed.

2.4. Algorithm Overview and Contributions

As we discussed at the start of this paper, all DPs that solve multiperiod lot-sizing problems require iterative computation, typically over time. For instance, to solve the single-stage capacitated problem defined in Florian and Klein (1971), one needs to solve the optimality equation for each state (that is, cumulative production quantity) in a given period, and then repeat these computations for each subsequent period in order to determine the optimal policy and resulting plan.

Even for our problem, it is possible to obtain the minimum cost of a regeneration network $\mathcal{N}$, $f(\mathcal{N})$, using the same approach (i.e., iterating over time period). However, as observed by van Hoesel et al. (2005), this approach cannot solve the problem in polynomial time, because the number of states that needs to be considered is exponential. In this paper, we propose a different way to conduct iterative computations to solve the MLSP-PC. Instead of iterating over time, we develop a DP-based algorithm to find $f(\mathcal{N})$ by iteratively solving the DP along a specially selected sequence of nodes (defined by time index and stage), which we call a basis path.

Our key algorithmic contribution is this basis path concept, and the bulk of this paper explains this concept and how it can be utilized to develop a polynomial algorithm for the MLSP-PC with general cost functions. This algorithm works in two phases: in the first phase, it computes the regeneration network costs $f(\mathcal{N})$, and in the second phase, it generates the entire production and distribution plan by choosing the best sequence of regeneration networks with the DP recursion (2). In addition, we develop an approach based on reducing this two-phase approach to a single phase, which we mention towards the end of the paper and detail in the appendix.

Below, we provide a conceptual overview of the algorithm, and in subsequent sections we formally present the details.

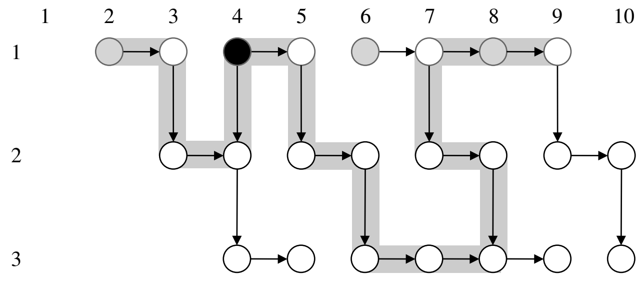
2.4.1. The Two-Phase Basis Path Algorithm. The key decisions in deterministic lot-sizing problems are production and transportation quantities—the inventory levels follow from these. Thus, algorithms for these problems are typically built around key properties related to optimal production, transportation, and inventory quantities. By reviewing the properties used in related lot-sizing problem, and demonstrating that these properties do not extend to our model, we can better contrast our algorithm from those that previously appeared in the literature.

The first (and arguably most important) property used to develop algorithms in the literature is the *Zero Inventory Ordering* (ZIO) property, which suggests that each production (transportation) quantity always be a sum of a sequence of demands. Every uncapacitated single-stage algorithm has utilized this ZIO property. The ZIO property generalizes to the *arborescent property* in multistage problems. In particular, the network flows corresponding to a production and distribution plan are arborescent, so that each node (i, j) is the root of a directed subtree of flows satisfying some subset of demands. Consequently, the production or transportation quantity at any node (i, j) is the sum of its immediate successor nodes' quantities, which are also set in a similar way by their immediate successor nodes, and so forth, leading to the leaf nodes whose quantities are given as a sum of successive demands. Consequently, each production or transportation quantity can ultimately be derived starting with a sum of successive demands. Zangwill (1969) solved the multistage uncapacitated problem with an algorithm that exploits this arborescent property.

Under the presence of capacity restrictions on production, however, the arborescent property is no longer valid. For the single-stage capacitated problem, for a given regeneration interval $\mathcal{J}$, the *fractional production property* applies, where each production quantity is one of $\{0, \epsilon_{\mathcal{J}}, C\}$ where $\epsilon_{\mathcal{J}}$ is the partial (fractional) production quantity derived from the demands of the interval $\mathcal{J}$ (Florian and Klein 1971). Similarly, in the MLSP-PC with nonspeculative costs (van Hoesel et al. 2005), capacitated production at the first stage restricts the production quantity by the fractional production property. That is, whenever a regeneration network (more precisely, a retailer's interval $\mathcal{J}$) is given, one of $\{0, \epsilon_{\mathcal{J}}, C\}$ will be the production quantity. On the other hand, each transportation quantity is determined by the arborescent property (as illustrated in Figure 6). That is, each transportation quantity is a sum of consecutive demands. Thus, the algorithm in van Hoesel et al. (2005) combines properties first identified in Florian and Klein (1971) and Zangwill (1969), leading to a fractional production policy and an arborescent transportation policy. In other words, as soon as production quantities are determined, a complete production and distribution plan can be efficiently generated using the arborescent property.

Under the general concave cost structure, we can determine production quantities using the fractional policy for each regeneration network. However, the distribution policies are no longer determined by manufacturing decisions. As illustrated in Figure 5, optimal flows for the general concave case may not be arborescent. Thus, one cannot explicitly determine transportation quantities given a production schedule. Consequently, results used to solve the aforementioned problems cannot solve the MLSP-PC with general concave costs because none of these previously used properties are relevant for making transportation decisions. To overcome this, we identify a unique path from the first

Figure 5. The basis path for regeneration network (2, 9, 4, 10).



node $(1, s_1)$ to the last node $(1, s_2)$ of the manufacturer's horizon in each regeneration network $\mathcal{N} = (s_1, s_2, t_1, t_2)$ (an example is illustrated in Figure 5). We then show that *below this basis path the arborescent property still applies*, and hence we can easily determine transportation decisions below the basis path. We then develop novel approaches to determining transportation policies above the basis path, so that we can complete the entire plan. Although there can be an exponential number of these basis paths, our algorithm doesn't explicitly enumerate all of these paths, but instead constructs the best basis path incrementally while simultaneously determining production and transportation quantities.

2.4.2. The Single-Phase Algorithm. With very few exceptions (e.g., van den Heuvel and Wagelmans 2006), algorithms for capacitated lot-sizing problems operate in two phases, first calculating the minimum cost subplan for all possible regeneration networks (regeneration intervals), and then selecting the sequence of subplans with minimum total cost. We are able to improve the complexity of the algorithm described above by developing an approach to simultaneously complete these two tasks; details can be found in Appendix S.3.

3. Basis Path, State, and Partial Trees

Before formally developing our DP approach, we introduce several key concepts in this section. We present the concept of a basis path and define the state variables of the value function in §3.1. In §3.2, we introduce the concept of partial trees, which are used to describe subplans, and in §3.3, we present the structural relationship between a basis path and partial trees. Subsection 3.4 defines the costs associated with partial trees.

3.1. Basis Path and State

Consider a regeneration network $\mathcal{N} = (s_1, s_2, t_1, t_2)$ of an extreme point solution (satisfying Proposition 1). Because it has no cycle, it is a tree, so there is a unique undirected *path* between any two nodes. In this context, a path from a node v_1 to another node v_k in $\mathcal{N}$ is a sequence of distinct nodes (i_r, j_r) such that consecutive nodes (i_r, j_r) and (i_{r+1}, j_{r+1}) have a flow between them for $r = 1, 2, \ldots, k-1$. We are particularly interested in the path between nodes $(1, s_1)$ and $(1, s_2)$ in the first stage, which we call the *basis path* of the regeneration network. We call each element of a basis path a *basis node*, and note that a basis path coincides with the traditional regeneration interval in the single-stage capacitated problems.

For a given basis path $\mathcal{P} = \{v_1, \ldots, v_k\}$ of the regeneration network $\mathcal{N}$, the DP evaluates production or transportation decisions for a state at a basis node and then moves to the next basis node. A state at a basis node v_r is represented by a triple (s, n_s, t), where n_s represents the total production quantity during the interval $[s+1, s_2]$ at the manufacturer, and $t+1$ is the starting time of an interval $[t+1, t_2]$ of the retailer's demands. We call the quantity n_s the *projected cumulative production quantity* (after period s) and the interval $[t+1, t_2]$ of demands the *projected set of demands*. The information, (s, n_s, t) is sufficient to construct the production and distribution plan for the n_s units from the basis node v_r, provided that subplans with respect to the basis nodes $v_1, v_2, \ldots, v_{r-1}$ (that is, the nodes on the basis path following v_r) are preprocessed. Subsequently, we will describe how we can characterize all subplans using partial trees.

Figure 5 shows the basis path $\{(1, 2), (1, 3), (2, 3), \ldots, (1, 9)\}$ of the regeneration network (2, 9, 4, 10). Observe that the basis path is not sequential in time. For instance, the nodes from $(1, 2)$ to $(2, 8)$ are in temporal order, but nodes $(2, 8)$ and $(2, 7)$ are not. In general, any DP, if it makes multiperiod decisions along a basis path, might move from a state to another state with or against the time index.

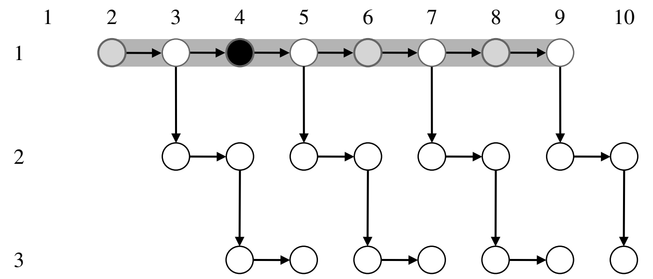
An example of a regeneration network with "nonspeculative" cost structure is given in Figure 6. Figures 5 and 6 highlight the differences between the MLSP-PC with general concave costs and the MLSP-PC with nonspeculative costs. For the nonspeculative cost structure, the basis path as we have defined it is explicitly ordered in time: $\mathcal{P} = \{(1, s_1), (1, s_1+1), \ldots, (1, s_2)\}$, and as we observed in the previous section, this is what allows van Hoesel et al. (2005) to solve the MLSP-PC by enumerating over the periods associated with the manufacturer's decision, something that cannot be done with general concave cost structures because the manufacturer's decisions are not sufficient to characterize the optimal policy.

Let $f_{\mathcal{N}}(\mathcal{P})$ be the minimum cost for a given basis path $\mathcal{P}$. Then, the cost $f(\mathcal{N})$ of the regeneration network is determined by solving the following problem.

$$f(\mathcal{N}) = \min_{\mathcal{P}} \{f_{\mathcal{N}}(\mathcal{P})\}. \tag{3}$$

To find an optimal solution for the regeneration network $\mathcal{N}$, we first find an extreme-point solution that achieves the minimum cost for each basis path $\mathcal{P}$. Then, to determine the optimal solution for the regeneration network $\mathcal{N}$, the optimal algorithm needs to determine which basis path is optimal and then the state of each node along the optimal basis path. In the single-stage problem CLSP, the basis path is the same as the regeneration interval, so it

Figure 6. The regeneration network and its basis path for nonspeculative costs.



suffices to determine production quantities. Unfortunately, in the multistage problem the basis path is more involved.

As mentioned above, this approach alone does not immediately lead to a polynomial-time algorithm (because the number of basis paths is exponential in T and L). However, by utilizing the structure of the regeneration network—specifically, each node has no more than four neighbors—we determine the cost at each basis node associated with neighboring basis nodes. This allows us to significantly reduce the number of paths to be considered, resulting in a polynomial algorithm.

In the next section, we present our algorithm to find the minimum cost for a given basis path, $f_{\mathcal{N}}(\mathcal{P})$, and explain how decisions along the basis path fully determine the complete production and distribution plan. First, we present preliminary results and definitions.

3.2. Partial Trees

A path from (i_1, j_1) to (i_k, j_k) is called a *manufacturer-retailer path* if the first node (i_1, j_1) is a manufacturer's node and the final node (i_k, j_k) is a retailer's node (i.e., $i_1 = 1$ and $i_k = L$). In a given regeneration network, any *partial tree* (or *subtree*) containing a manufacturer-retailer path is called a *comprehensive tree* and the partial trees that have no manufacturer-retailer path are called *noncomprehensive trees*.

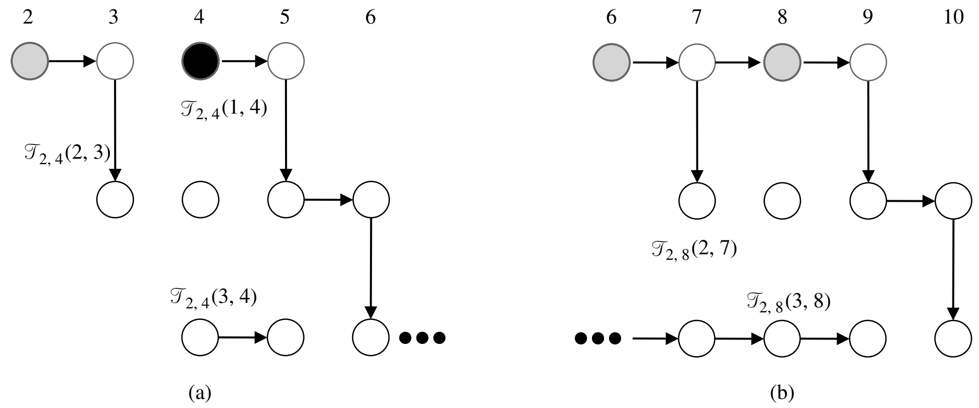
Figure 5 illustrates these definitions. The subtree consisting of nodes $(1,2)$, $(1,3)$, and $(2,3)$ is a noncomprehensive tree, but the expanded subtree with nodes $(1,2)$, $(1,3)$, $(2,3)$, $(2,4)$, and $(3,4)$ is a comprehensive tree. If we remove the basis path from the regeneration network $(2,9,4,10)$ in Figure 5, the remaining components are all noncomprehensive trees, which we specifically call *dangling trees* with respect to the basis path. The dangling trees above and below the basis path are referred to as *upper* and *lower* dangling trees, respectively. Our algorithm will carry out computations along the basis path, incorporating the costs associated with dangling trees along the way.

Given a node $v = (i, j)$, it is convenient to identify its *neighborhood* nodes to the north, south, east, and west of the node v: $(i-1, j)$, $(i+1, j)$, $(i, j+1)$, and $(i, j-1)$. We define $B(v) = \{(i-1, j), (i+1, j), (i, j+1), (i, j-1)\}$ to be the set of neighborhood nodes of (i, j). For notational consistency, we add dummy nodes $(0, j)$, $(L+1, j)$, $(i, 0)$, and $(i, T+1)$ for $1 \leqslant i \leqslant L$ and $1 \leqslant j \leqslant L$ so that any node (i, j) always has four neighbors. Consider the connected component of $\mathcal{N}$, which includes the basis path, in which every node has flows into/out of other nodes. Because the regeneration network $\mathcal{N}$ is a tree (because $\mathcal{N}$ has no cycle), cutting any arc of flow divides the component into two parts. For instance, if we remove the arc of a flow between nodes (i, j) and $(i-1, j)$, the regeneration network is divided into two subtrees. One of them, which includes the north node $(i-1, j)$, is denoted by $\mathcal{T}_{ij}(i-1, j)$. In a similar way, we define the south, east, and west trees $\mathcal{T}_{ij}(i+1, j)$, $\mathcal{T}_{ij}(i, j+1)$, and $\mathcal{T}_{ij}(i, j-1)$, respectively.

Figure 7(a) shows subtrees of node $(2,4)$ in the regeneration network described in Figure 5, which are generated by cutting the arcs of flow around $(2,4)$. Because there is no flow to the east of node $(2,4)$, i.e., from node $(2,4)$ into node $(2,5)$, we say that the east tree is empty. Note that trees $\mathcal{T}_{2,4}(3,4)$ and $\mathcal{T}_{2,4}(2,3)$ are noncomprehensive trees, whereas $\mathcal{T}_{2,4}(1,4)$ is a comprehensive tree. Figure 7(b) shows comprehensive trees $\mathcal{T}_{2,8}(2,7)$ on the west and $\mathcal{T}_{2,8}(3,8)$ on the south of node $(2,8)$ where the north and east trees are empty. Figure 7 also shows that node $(2,4)$ has only one comprehensive tree around it, whereas node $(2,8)$ has two comprehensive trees.

When determining $f_{\mathcal{N}}(\mathcal{P})$ and subsequently solving the problem defined in Equation (3), the cost associated with each transition (from the current state at node v_r to the next state at node v_{r+1}) depends on the location of dangling trees around the current basis node. In addition to the fact that upper (lower) dangling trees are located above (below) the basis path, we need more precise information about the location (north, south, east, or west) of dangling trees in order to determine the associated production/transportation and inventory costs. To this end, we next explore the relationship among basis nodes, comprehensive trees, and dangling trees.

Figure 7. Trees around node $(2,4)$ and node $(2,8)$.



3.3. Structure of Partial Trees Around a Basis Node

Although there are many possible basis paths connecting the two nodes $v_1 = (1, s_1)$ and $v_k = (1, s_2)$ in the manufacturer's stage of the regeneration network $\mathcal{N} = (s_1, s_2, t_1, t_2)$, we can restrict our attention to special cases by utilizing the property that $\mathcal{N}$ contains no cycles. We begin by bounding the number of comprehensive trees around each basis node. For this, we use the no-cycle property (Proposition 1) and the fact that a comprehensive tree always contains a manufacturer-retailer path (from the definition).

Proposition 2. *Given a node v on a regeneration network, there are at most two comprehensive trees among the four trees* $\mathcal{T}_v(u)$, $u \in B(v)$.

For each basis node v_r in the basis path $\mathcal{P} = \{v_1, \ldots, v_k\}$, we call nodes v_{r-1} and v_{r+1} the *previous node* and *subsequent node* of v_r, respectively. For notational convenience, we assume that the first node v_1 has as its previous node $v_0 = (1, s_1 - 1)$, and the last node v_k has its subsequent node $v_{k+1} = (1, s_2 + 1)$. In the same manner, we call $\mathcal{T}_{v_r}(v_{r-1})$ and $\mathcal{T}_{v_r}(v_{r+1})$ the *previous* and *subsequent trees* of basis node v_r. We note that the previous tree contains nodes $v_1, v_2, \ldots, v_{r-1}$ and the subsequent tree contains nodes $v_{r+1}, v_{r+2}, \ldots, v_k$, and we also notice that all subsequent trees are comprehensive. To see this, first consider the subsequent tree $\mathcal{T}_{v_{k-1}}(v_k)$ of node v_{k-1}. Because node v_k, i.e., the manufacturer's last node $(1, s_2)$, has a path to the retailer's last node (L, t_2), we see that the tree $\mathcal{T}_{v_{k-1}}(v_k)$ is a comprehensive tree. Then, any subsequent tree $\mathcal{T}_{v_r}(v_{r+1})$ of basis node v_r, $r < k$, contains the node v_k, and hence it has a manufacturer-retailer path from v_k to (L, t_2), implying that it must be a comprehensive tree. However, previous trees are not always comprehensive trees. For example, the previous tree $\mathcal{T}_{v_2}(v_1)$ of node v_2 is not comprehensive.

For given previous and subsequent nodes, v_{r-1} and v_{r+1}, we can determine the locations and (either upper or lower) types of dangling trees around a basis node v_r. To see this, suppose that the subsequent node is north of v_r, i.e., $v_{r+1} = (i-1, j)$. Then, the previous node will be one of $(i, j+1)$, $(i+1, j)$, and $(i, j-1)$. The three possible cases for the location of the previous tree when $v_{r+1} = (i-1, j)$ are illustrated in Figure 8(a). The possible cases when $v_{r+1} = (i, j+1)$ are illustrated in Figure 8(b). Note that when the subsequent node lies to the south of node v_r (i.e., $v_r = (i+1, j)$), there are only two possible locations for the previous node as only acyclic subnetworks are permitted. Specifically, the previous node cannot be the east node $(i, j+1)$ because we cannot have the east comprehensive tree $\mathcal{T}_{ij}(i, j+1)$ preceding the south comprehensive tree $\mathcal{T}_{ij}(i+1, j)$ without creating a cycle. Likewise, when $v_{r+1} = (i, j-1)$, the previous node cannot be the north node $(i-1, j)$ (see Figures 8(c) and (d), respectively). Figure 8 illustrates all possible 10 cases. Note that there can be at most two dangling trees at any basis node. Also note that their locations (that is, their directions) are determined once the previous and subsequent trees are identified. For instance, if the three consecutive basis nodes are $v_{r-1} = (i+1, j)$, $v_r = (i, j)$, $v_{r+1} = (i-1, j)$, there can be an upper dangling tree on the west and/or a lower dangling tree on the east (Figure 8(a2)). If the three nodes are $(i+1, j)$, (i, j), and $(i, j+1)$, only upper dangling trees can exist (at the north and west of node (i, j)).

Before closing this subsection, it is worth emphasizing the relationship between the state (s, n_s, t) at the current basis node, v_r and the subsequent tree, $\mathcal{T}_{v_r}(v_{r+1})$. A state (s, n_s, t) of basis node v_r implies a minimum-cost subsequent tree containing the basis nodes $v_{r+1}, v_{r+2}, \ldots, v_k$. As a result, $\mathcal{T}_{v_r}(v_{r+1})$ has the manufacturer's nodes for periods $s+1$ through s_2, during which n_s units are produced and allocated for demands $d_{t+1}, d_{t+2}, \ldots, d_{t_2}$.

3.4. Costs of Dangling Trees

For upper dangling trees (of basis node (i, j)), we define $\varphi(i, j)^{a, s', s}$ to be the minimum cost to produce $a \in \Omega_{\mathcal{J}}$ units during $[s', s]$ in the manufacturer's horizon and then to transport/inventory these units to intermediate node (i, j), $s' \leqslant s \leqslant j$, $1 \leqslant i \leqslant L$. If $s' > s$, we set $\varphi(i, j)^{a, s', s} = 0$. If a basis node (i, j) lies in the manufacturer's horizon, i.e., $i = 1$, it has no upper dangling trees. For convenience, we define a virtual upper tree $\mathcal{T}_{1, j}(0, j)$ of node $(1, j)$ so we can consistently use the term $\varphi(i, j)^{a, s', s}$; we define $\varphi(0, j)^{a, s', s} = 0$ for any arguments a, s' and s.

For lower dangling trees, we define $\phi(i, j)_{t', t}$ to be the minimum cost of satisfying demands $d_{t'}, d_{t'+1}, \ldots, d_t$ using $d_{[t', t]}$ units in node (i, j), $j \leqslant t' \leqslant t$. If $t' > t$, we set $\phi(i, j)_{t', t} = 0$. We also define a virtual lower dangling tree, for convenience, for basis nodes in the retailer's horizon. We define $\phi(L+1, j)_{t', t} = 0$ for any arguments t' and t. Given these definitions, the procedure to compute the costs of upper and lower dangling trees is relatively straightforward because of their arborescent structures. For completeness, we provide details in Appendix S.1.

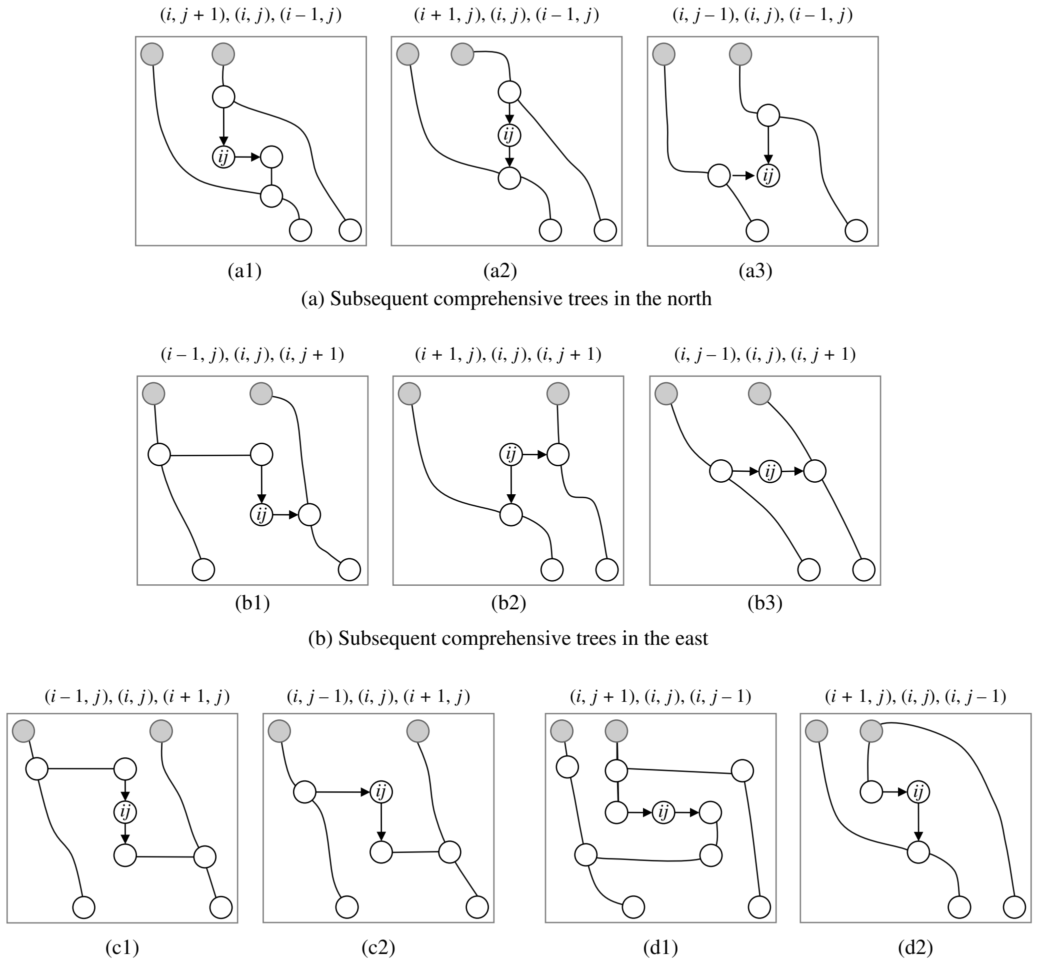
4. Planning with Known Basis Path

In this section, we assume that we are given a basis path $\mathcal{P}$ over which our algorithm iterates. To illustrate how our algorithm works, we first start with the single-stage problem CLSP in which each state can be described by a projected cumulative quantity and then extend it to the multistage problem in which each state needs a projected set of demands as well as a projected cumulative quantity. We observe that the approach for solving the CLSP in the next subsection is substantially different from Florian and Klein (1971) in the sense that they use as a state the cumulative production quantity (rather than projected cumulative production quantity).

4.1. Planning with Projected Cumulative Production Quantities

Note that in the CLSP, $s_1 = t_1$ and $s_2 = t_2$ for any regeneration network $\mathcal{N} = (s_1, s_2, t_1, t_2)$. That is, the retailer's

Figure 8. Possible forms of comprehensive trees around (i, j).


(i, j + 1), (i, j), (i – 1, j)
(i + 1, j), (i, j), (i – 1, j)
(i, j – 1), (i, j), (i – 1, j)
ij
(a1)
(a2)
(a3)
(i – 1, j), (i, j), (i, j + 1)
(i + 1, j), (i, j), (i, j + 1)
(i, j – 1), (i, j), (i, j + 1)
(b1)
(b2)
(b3)
(i – 1, j), (i, j), (i + 1, j)
(i, j – 1), (i, j), (i + 1, j)
(i, j + 1), (i, j), (i, j – 1)
(i + 1, j), (i, j), (i, j – 1)
(c1)
(c2)
(d1)
(d2)

(a) Subsequent comprehensive trees in the north

(b) Subsequent comprehensive trees in the east

(c) Subsequent comprehensive trees in the south

(d) Subsequent comprehensive trees in the west

interval, $\mathcal{I} = [t_1, t_2]$ coincides with the production interval, $[s_1, s_2]$. As a result, any regeneration network $\mathcal{N}$ has only one basis path $\{(1, s_1), \dots, (1, s_2)\}$, which is a regeneration interval in the usual sense.

For each node $(1, s)$, $s \in \mathcal{I}$, the projected cumulative production quantity n_s after period $s+1$ is $n_s = x_{1,s+1} + \cdots + x_{1,s_2}$ where $n_s \in \Omega_{\mathcal{I}}$. These quantities, n_s for $s = s_1, \dots, s_2$, are sufficient to compute the detailed production plan. We begin by denoting by (s, n_s) the state in which we produce n_s units during $[s+1, s_2]$ to satisfy demands in $[s+1, s_2]$. In period s, we must determine the number of units to be produced, which must be one of zero, partial production quantity $\epsilon_{\mathcal{I}}$, or full production quantity C (Proposition 1). If the production quantity in period s is $a \in \{0, \epsilon_{\mathcal{I}}, C\}$, then the state at period $s-1$ is described by $(s-1, a+n_s)$ where $a + n_s \in \Omega_{\mathcal{I}}$.

To compute the cost of regeneration network $\mathcal{N}$, let $f_{\mathcal{N}}(s)^{n_s}$ be the minimum cost of satisfying the demands in the interval $[s_1, s] \subseteq \mathcal{I}$ when the projected cumulative production quantity is n_s. (We note that the state of projected cumulative production appears in superscript. We will also follow this convention when dealing with the multistage problem.)

Here, the quantity n_s is not arbitrary, but is instead one of the values in $\Omega_{\mathcal{I}}$ (and indeed, this is what makes the problem polynomially solvable). Note that the cost $f(\mathcal{N})$ of the regeneration network is equal to $f_{\mathcal{N}}(s_2)^0$. To obtain $f_{\mathcal{N}}(s)^{n_s}$, in general we only need to determine the production quantity a in period s. Because period s_2 is a regeneration period such that $I_{1,s_2} = 0$, the cumulative production quantity from period s through s_2 is not larger than the total sum of demands $d_{[s,s_2]}$, i.e., $a + n_s \leqslant d_{[s,s_2]}$. Then, the additional $d_{[s,s_2]} - a - n_s$ units must be held in inventory at the end of period $s-1$ to meet remaining demand that production during $[s, s_2]$ could not supply for the demands $d_s, \dots, d_{s_2}$. Let $c_{\mathcal{I}}(s)^{a,s,n_s}$ be the immediate cost at the current node $(1, s)$, which consists of two components: the production cost in period s, and the cost of carrying the inventory at the end of period $s-1$, i.e.,

$$c_{\mathcal{I}}(s)^{a,s,n_s} = h_{1,s-1}(d_{[s,s_2]} - a - n_s) + p_{1,s}(a).$$

The cost $c_{\mathcal{I}}(s)^{a,s,n_s}$ at node $(1, s)$ is in fact the cost from changing state $(s-1, a+n_s)$ to state (s, n_s). With the immediate costs, we can compute the minimum cost $f_{\mathcal{N}}(s)^{n_s}$ using the following optimality equation:

$$f_{\mathcal{N}}(s)^{n_s} = \min_a \{ f_{\mathcal{N}}(s-1)^{a+n_s} + c_{\mathcal{J}}(s)^{a,s,n_s} \mid a + n_s \in \Omega_{\mathcal{J}} \}, \quad (4)$$

with initial condition $f_{\mathcal{N}}(s_0)^{d_{\mathcal{J}}} = 0$ and $s_0 = s_1 - 1$. We note that the initial condition of the formula $(f_{\mathcal{N}}(s_0)^{d_{\mathcal{J}}} = 0)$ means that the projected cumulative quantity is just the total sum of demands that is to be produced after period s_0, and hence there is nothing to be done at period s_0.

4.2. Immediate Cost at a Basis Node for the MLSP-PC

The recursive formula (4) demonstrates that the single-stage problem is solved by a forward algorithm in time periods. We solve the multistage problem using a similar approach. However, in contrast to the single-stage problem in which each basis path is the regeneration interval itself, the regeneration network in the multistage problem has basis paths that are not ordered in time period. We will evaluate the cost $f_{\mathcal{N}}(\mathcal{P})$ of an optimal policy for a given regeneration network with the basis path $\mathcal{P} = \{v_1, v_2, \ldots, v_k\}$ by extending the value function $f_{\mathcal{N}}(s)^{n_s}$ for the single-stage problem (CLSP) to the multistage problem (MLSP-PC). To do this, we need to define the immediate cost at a basis node v_r.

For a regeneration network $\mathcal{N}$ with retailer's interval $\mathcal{J} = [t_1, t_2]$, consider a basis node v_r with its state (s, n_s, t) and its previous and subsequent nodes v_{r-1} and v_{r+1}. By construction, the subsequent tree $\mathcal{T}_{v_r}(v_{r+1})$ contains manufacturer's nodes $s+1$ through s_2 producing n_s units, and retailer's nodes $t+1$ through t_2. Suppose that node v_r has upper dangling trees containing manufacturer's periods $s'+1$ through s and lower dangling trees containing retailer's periods $t'+1$ through t where $s_1 \leqslant s' \leqslant s \leqslant s_2$ and $t_1 \leqslant t' \leqslant t \leqslant t_2$. Let a be the total production quantity during $[s'+1, s]$ of the upper dangling trees. Then we define the immediate cost as follows:

Definition 1. The immediate cost at basis node v_r, denoted by $c_{\mathcal{J}}(v_{r-1}, v_r, v_{r+1})_{t',t}^{s',a,s,n_s}$, is the minimum cost associated with the flows between v_r and v_{r-1}, the flows from all upper dangling trees into v_r, and the flows from v_r to all lower dangling trees.

Note that the immediate cost at a basis node v_r includes all costs associated with dangling trees and the flows around node v_r except for the cost of the flow between nodes v_r and v_{r+1} (i.e., the cost of the flow between the current and subsequent nodes), which is considered when the immediate cost at basis node v_{r+1} is determined.

The precise functional form of the immediate cost depends on the types of dangling trees and their locations (i.e., the 10 cases listed in Figure 8). Because a basis node can have up to two dangling trees, some possibilities exist: both trees are upper dangling trees, both trees are lower dangling trees, one of them is an upper dangling tree and the other one is a lower dangling tree, or there is only one (or no) dangling tree (see Figure 8). Any case with one or no dangling tree is a special case of two dangling trees.

When determining $c_{\mathcal{J}}(v_{r-1}, v_r, v_{r+1})_{t',t}^{s',a,s,n_s}$ (we abbreviate this to $c_{\mathcal{J}}(\cdot)$ where the meaning is obvious), we note that we only need information about the retailer's interval $\mathcal{J} = [t_1, t_2]$, not the full information about a regeneration network $\mathcal{N} = (s_1, s_2, t_1, t_2)$. This is because the (partial) production quantity depends only on retailer's interval $\mathcal{J}$. In other words, the quantity a in the cost $c_{\mathcal{J}}(v_{r-1}, v_r, v_{r+1})_{t',t}^{s',a,s,n_s}$ should be one of $\{0, \epsilon_{\mathcal{J}}, C\}$.

In Appendix S.2, we detail how to determine the immediate costs for two representative cases. The first case has one upper dangling tree and one lower dangling tree, with three consecutive basis nodes $(i+1, j)$, (i, j) and $(i-1, j)$. The second case has two upper dangling trees with three consecutive basis nodes $(i+1, j)$ (i, j) and $(i, j+1)$. (Figure 8). All remaining cases (there are 10 cases as shown in Figure 8) can be analyzed in a similar manner. Below, we show that the second case (where v_r has two upper dangling trees) is the most computationally demanding, and thus determines the complexity of algorithm.

Applying the logic developed in Appendix S.2, we can compute the immediate costs for all of the 10 cases in Figure 8 (see Table 1). To clarify which parameters (and information) are necessary to compute $c_{\mathcal{J}}(\cdot)$, we omit unnecessary arguments in the third column of Table 1. For example, when $v_{r-1} = (i-1, j)$, $v_r = (i, j)$ and $v_{r+1} = (i+1, j)$, we do not have any dangling trees (the dangling trees are all empty). In this case, if our current state is (s, n_s, t) at node v_r, we have $s' = s$, $a = 0$, and $t' = t$ in $c_{\mathcal{J}}(v_{r-1}, v_r, v_{r+1})_{t',t}^{s',a,s,n_s}$. Omitting the (trivial) terms s' and a when the upper dangling tree is empty (or the t' when the lower dangling tree is empty) clarifies the presentation.

We now analyze the computational complexity of determining costs $c_{\mathcal{J}}(v_{r-1}, v_r, v_{r+1})_{t',t}^{s',a,s,n_s}$ for a given regeneration network with retailer's time interval $\mathcal{J}$. First, note that the number of cases of consecutive basis nodes (v_{r-1}, v_r and v_{r+1}) is $O(LT)$. This is because the number of nodes v_r is $O(LT)$ and because any node v_r will have its previous (subsequent) node v_{r-1} (v_{r+1}) as one of the four components of $B(v_r)$. We focus on the complexity of computing the immediate cost for possible states. The second column of Table 1 presents the immediate costs for all the 10 cases. It shows that only 2 cases involve a minimum operator: the case with two lower dangling trees, $(v_{r-1} = (i, j-1)$, $v_r = (i, j)$, $v_{r+1} = (i-1, j))$ or the case with two upper dangling trees $(v_{r-1} = (i+1, j)$, $v_r = (i, j)$, $v_{r+1} = (i, j+1))$. The remaining 8 cases can be evaluated in constant time.

To determine the complexity of these eight cases, first note that the computing time depends on the number of arguments necessary to determine $c_{\mathcal{J}}(v_{r-1}, v_r, v_{r+1})_{t',t}^{s',a,s,n_s}$. The necessary arguments are shown in the third column of Table 1. Observe that the two worst cases among these eight cases are $((i+1, j), (i, j), (i-1, j))$ and $((i, j-1),$

Table 1. Immediate costs at basis nodes.

Basis nodes	Cost at basis node (i, j)	$c_{\mathcal{J}}(\cdot)_{t',t}^{s',a,s,n_s}$
$(i, j+1), (i, j), (i-1, j)$	$h_{ij}(a+n_s-d_{[t+1,t_2]})+\varphi(i, j-1)^{a,s'+1,s}+h_{i,j-1}(a)$	$c_{\mathcal{J}}(\cdot)_{t}^{s',a,s,n_s}$
$(i+1, j), (i, j), (i-1, j)$	$p_{i+1,j}(a+n_s-d_{[t'+1,t_2]})+\phi(i, j+1)_{t'+1,t}+h_{ij}(d_{[t'+1,t]})$ $\varphi(i, j-1)^{a,s'+1,s}+h_{i,j-1}(a)$	$c_{\mathcal{J}}(\cdot)_{t',t}^{s',a,s,n_s}$
$(i, j-1), (i, j), (i-1, j)$	$\min^1\{h_{i,j-1}(d_{[t'+1,t_2]}-n_s)+\phi(i+1, j)_{t'+1,t''}+p_{i+1,j}(d_{[t'+1,t'']})$ $+\phi(i, j+1)_{t''+1,t}+h_{i,j}(d_{[t''+1,t]})\}$	$c_{\mathcal{J}}(\cdot)_{t',t}^{s,n_s}$
$(i-1, j), (i, j), (i, j+1)$	$p_{ij}(d_{[t'+1,t_2]}-n_s)+\phi(i+1, j)_{t'+1,t}+p_{i+1,j}(d_{[t'+1,t]})$	$c_{\mathcal{J}}(\cdot)_{t',t}^{s,n_s}$
$(i+1, j), (i, j), (i, j+1)$	$\min^2\{p_{i+1,j}(a+n_s-d_{[t+1,t_2]})+\varphi(i, j-1)^{a-b,s'+1,s''}$ $+h_{i,j-1}(a-b)+\varphi(i-1, j)^{b,s''+1,s}+p_{ij}(b)\}$	$c_{\mathcal{J}}(\cdot)_{t}^{s',a,s,n_s}$
$(i, j-1), (i, j), (i, j+1)$	$h_{i,j-1}(d_{[t'+1,t_2]}-a-n_s)+\phi(i+1, j)_{t'+1,t}+p_{i+1,j}(d_{[t'+1,t]})$ $+\varphi(i-1, j)^{a,s'+1,s}+p_{ij}(a)$	$c_{\mathcal{J}}(\cdot)_{t',t}^{s',a,s,n_s}$
$(i-1, j), (i, j), (i+1, j)$	$p_{ij}(d_{[t+1,t_2]}-n_s)$	$c_{\mathcal{J}}(\cdot)_{t}^{s,n_s}$
$(i, j-1), (i, j), (i+1, j)$	$h_{i,j-1}(d_{[t+1,t_2]}-a-n_s)+\varphi(i-1, j)^{a,s'+1,s}+p_{ij}(a)$	$c_{\mathcal{J}}(\cdot)_{t}^{s',a,s,n_s}$
$(i, j+1), (i, j), (i, j-1)$	$h_{ij}(n_s-d_{[t+1,t_2]})$	$c_{\mathcal{J}}(\cdot)_{t}^{s,n_s}$
$(i+1, j), (i, j), (i, j-1)$	$p_{i+1,j}(n_s-d_{[t'+1,t_2]})+\phi(i, j+1)_{t'+1,t}+h_{ij}(d_{[t'+1,t]})$	$c_{\mathcal{J}}(\cdot)_{t',t}^{s,n_s}$

$\min^1$ is over $t' \leqslant t'' \leqslant t$.
$\min^2$ is over $b \in \Omega_{\mathcal{J}}, s' \leqslant s'' \leqslant s$.

$(i, j), (i, j+1))$: both cases have six arguments: s', a, s, n_s, t' and t. Because each of these arguments has $O(T)$ possible instances, for given (v_{r-1}, v_r, v_{r+1}), the maximum complexity is $O(T^6)$. Because there are $O(LT)$ basis nodes to be considered, the total complexity is $O(LT^7)$.

We now consider the two cases that contain a minimum operator. In the case with two upper dangling trees, the expression inside the minimum operator has five arguments ($O(T^5)$). To determine $c_{\mathcal{J}}(v_{r-1}, v_r, v_{r+1})_{t',t}^{s',a,s,n_s}$, we need to enumerate over b and s'' ($O(T^2)$). Hence, it takes $O(T^7)$ time to determine $c_{\mathcal{J}}(v_{r-1}, v_r, v_{r+1})_{t',t}^{s',a,s,n_s}$ for given (v_{r-1}, v_r, v_{r+1}). Because there are $O(LT)$ basis nodes, we can see that the total computing time for the case with two upper dangling trees is $O(LT^8)$. A similar analysis shows that it takes $O(LT^6)$ for the case with two lower dangling trees.

Comparing the complexities of all 10 cases, we conclude that it takes $O(LT^8)$ to compute all the immediate costs for a given regeneration network with retailer's time interval $\mathcal{J}$. Finally, because there are $O(T^2)$ intervals $\mathcal{J}$, the total computing time for all regeneration networks is $O(LT^{10})$. Appendix S.3 describes several ways to further reduce the complexity.

4.3. An Algorithm with Known Basis Path

Given the immediate cost at each basis node, we can compute the optimal cost along the basis path $\mathcal{P} = \{v_1, \ldots, v_k\}$ in the regeneration network $\mathcal{N} = (s_1, s_2, t_1, t_2)$ with retailer's interval $\mathcal{J} = [t_1, t_2]$ where $v_1 = (1, s_1)$ and $v_k = (1, s_2)$. If $\mathcal{P}$ contains a single node, i.e., $s_1 = s_2$, then we satisfy every demand d_t, $t \in \mathcal{J}$, by a single production in period s_1. This is exactly the case of the uncapacitated multistage problem, which can be solved by Zangwill's algorithm based on the arborescent tree (or lower dangling tree) structure: $f_{\mathcal{N}}(\mathcal{P}) = p_{1,s_1}(d_{\mathcal{J}}) + \phi(1, s_1)_{t_1,t_2}$. Rather than build a "case-by-case" algorithm, however, we take the following approach. Recall that the previous node of v_1 is $v_0 = (1, s_1 - 1)$ and the subsequent node of v_k is $v_{k+1} = (1, s_2 + 1)$; $s_0 = s_1 - 1$ and $t_0 = t_1 - 1$. The cost of $\mathcal{N}$ with a single node v_1 thus equals $c_{\mathcal{J}}(v_0, v_1, v_2)_{t_0,t_2}^{s_0,d_{\mathcal{J}},s_2,0}$. If the basis path has multiple nodes, we need to define the following function (analogous to $f_{\mathcal{N}}(s)^{n_s}$ in the CLSP problem) in order to solve the dynamic program $f_{\mathcal{N}}(\mathcal{P})$.

Definition 2. Let $f_{\mathcal{P},\mathcal{N}}(v_r)_t^{s,n_s}$ be the minimum cost of satisfying $d_{t_1}, d_{t_1+1}, \ldots, d_t$ for a regeneration network $\mathcal{N}$ with basis path $\mathcal{P}$ when the state at the current basis node, v_r, is (s, n_s, t).

We can therefore represent the cost $f_{\mathcal{N}}(\mathcal{P})$ of the network $\mathcal{N}$ with respect to the basis path $\mathcal{P}$ as $f_{\mathcal{P},\mathcal{N}}(v_k)_{t_2}^{s_2,0}$. We can obtain this value using a recursion similar to that developed in §4.1 for the CLSP. As an initial condition, we set $f_{\mathcal{P},\mathcal{N}}(v_0)_{t_0}^{s_0,d_{\mathcal{J}}} = 0$ and iterate along the basis path. Then, the following recursive equations determine the optimal plan for a given basis path $\mathcal{P}$ in regeneration network $\mathcal{N}$:

$$f_{\mathcal{P},\mathcal{N}}(v_0)_{t_0}^{s_0,d_{\mathcal{J}}} = 0,$$

$$f_{\mathcal{P},\mathcal{N}}(v_r)_t^{s,n_s} = \min_{\substack{a \\ s_0 \leqslant s' \leqslant s \\ t_0 \leqslant t' \leqslant t,\, s' \leqslant t'}} \Big\{ f_{\mathcal{P},\mathcal{N}}(v_{r-1})_{t'}^{s',a+n_s} + c_{\mathcal{J}}(v_{r-1}, v_r, v_{r+1})_{t',t}^{s',a,s,n_s} \,\big|\, a+n_s \in \Omega_{\mathcal{J}} \Big\}. \quad (5)$$

Given the optimal cost $f_{\mathcal{P},\mathcal{N}}(v_k)_{t_2}^{s_2,0}$, we can use the set of optimal states to determine the subplan of upper and lower dangling trees, and thus a complete production and distribution plan for a given $\mathcal{N}$ and $\mathcal{P}$. Unfortunately, because the number of basis paths in a regeneration network is in general exponential, complete enumeration over all possible $\mathcal{P}$ will not result in a polynomial algorithm. However, as we detail in the next section, a slight modification of the procedure does in fact result in a polynomial-time algorithm.

5. A Polynomial Optimal Algorithm for the MLSP-PC

Observe that most computations in (5) are related to the immediate cost $c_{\mathcal{J}}(v_{r-1}, v_r, v_{r+1})_{t',t}^{s',a,s,n_s}$, which depends on three consecutive basis nodes but not on the entire path. This dependence on just three nodes makes a polynomial-time algorithm possible even when the basis path is not known.

Given a regeneration network $\mathcal{N} = (s_1, s_2, t_1, t_2)$, we know the first and last nodes of an optimal basis path; that is, $v_1 = (1, s_1)$ and $v_k = (1, s_2)$ for some k. (Note that to be consistent with our development to this point, we use v_k to denote the last node in the basis path.) The intermediate nodes (between v_1 and v_k) will be determined dynamically during the iteration of the DP. Because the basis nodes are determined by the DP, we need to extend the state (the projected quantity and the projected set of demands) for the previous algorithm with known basis path to include the current node v and its subsequent node w. Thus, we define a state by (v, w, s, n_s, t) that describes the situation in which v and w are assumed to be consecutive basis nodes (in a basis path), and the subsequent tree $\mathcal{T}_v(w)$ of node v produces $n_s \in \Omega_{\mathcal{J}}$ units during $[s+1, s_2]$ to meet demand in the interval $[t+1, t_2]$. Notice that if the current node v and its subsequent node w are known, the DP must identify the previous node to compute the immediate cost at v. Although we do not explicitly know its previous node, we know that it belongs to the neighborhood of v, i.e., $B(v)$. Because the subsequent node also belongs to $B(v)$, the previous node belongs more precisely to $B(v) - \{w\}$. For the node v and its subsequent node w, the optimal path from v back to the first node v_1 is obtained recursively by choosing the best neighbor node $u \in B(v) - \{w\}$. We explicitly define the value function for the state (v, w, s, n_s, t) below:

Definition 3. Let $f_{\mathcal{N}}(v, w)_t^{s,n_s}$ be the value function for the state (v, w, s, n_s, t); that is, the minimum cost of satisfying demands $d_{t_1}, d_{t_1+1}, \ldots, d_t$ if v and w are consecutive basis nodes in a basis path and the subsequent tree $\mathcal{T}_v(w)$ of node v produces $n_s \in \Omega_{\mathcal{J}}$ units during $[s+1, s_2]$ to meet demand in the interval $[t+1, t_2]$.

By this definition, the cost $f(\mathcal{N})$ of the regeneration network is $f_{\mathcal{N}}(v_k, v_{k+1})_{t_2}^{s_2,0}$. For the initial condition, we set $f_{\mathcal{N}}(v_0, v_1)_{t_0}^{s_0,d_{\mathcal{J}}} = 0$. From (5) developed for the case with a known basis path, we see that if the previous node of v is u, then for a given a, s', and t', the total cost is given as follows:

$$f_{\mathcal{N}}(u, v)_{t'}^{s',a+n_s} + c_{\mathcal{J}}(u, v, w)_{t',t}^{s',a,s,n_s}\colon\ a + n_s \in \Omega_{\mathcal{J}}.$$

By considering all cases of $u \in B(v) - \{w\}$, we obtain the following optimality equation:

$$f_{\mathcal{N}}(v_0, v_1)_{t_0}^{s_0,d_{\mathcal{J}}} = 0,$$
$$f_{\mathcal{N}}(v, w)_t^{s,n_s} = \min_{\substack{a \\ s_0 \leqslant s' \leqslant s \\ t_0 \leqslant t' \leqslant t,\, s \leqslant t'}} \big\{ f_{\mathcal{N}}(u, v)_{t'}^{s',a+n_s} + c_{\mathcal{J}}(u, v, w)_{t',t}^{s',a,s,n_s} \mid a + n_s \in \Omega_{\mathcal{J}}, u \in B(v) - \{w\} \big\}. \quad (6)$$

By solving this DP, we obtain the optimal policy for each state (v, w, s, n_s, t). The policy is described by, among other things, the sequence of (basis) nodes from the first node v_1 to the node v under consideration. Note that the path of nodes from v_1 to v is not unique, but depends on the subsequent node, the projected cumulative production quantity, and the projected set of demands with respect to the node v. Given a node v, the subsequent node w is one of the neighbors in $B(v)$, so that the possible number of the subsequent nodes is four. Because there are $O(T^2)$ pairs (s, n_s) for the projected cumulative production quantity and $O(T)$ of t for the projected set of demands, the total number of possible states with respect to node v is $O(4T^3)$. This means that there can be up to $O(T^3)$ basis paths from v_1 to v. Because there are $O(LT)$ nodes v, the total number of possible paths constructed during the DP iterations is $O(LT^4)$. That is, the number of paths is a polynomial function of L and T, which is determined by the size of the state space of (v, w, s, n_s, t).

To evaluate the overall complexity of this DP, first observe that each regeneration network $\mathcal{N} = (s_1, s_2, t_1, t_2)$ is specified by four parameters, so $O(T^4)$ regeneration networks are considered. Given a regeneration network $\mathcal{N}$, for each state (v, w, s, n_s, t) we evaluate $f_{\mathcal{N}}(v, w)_t^{s,n_s}$ for which we need additional parameters s', t', and the quantity a in the main optimality recursion (6). Hence, the overall complexity would seem to naively require up to $O(LT^{11})$. In Appendix S.3, we detail how eliminating double counting will reduce the complexity of this algorithm to $O(LT^{10})$, and how taking advantage of the structure of the cost function and consolidating regeneration networks will further reduce the complexity to $O(LT^8)$.

6. Conclusion

In this paper, we consider a multistage lot-sizing problem for a serial supply chain where the production is capacitated at the first stage. To capture the general economies of scale not only in production but also in transportation, we assume general concave costs with speculative cost structure through the entire supply chain. This general problem is significantly different from the special case disallowing speculative motives in transportation in that the manufacturer's decision at the first stage does not uniquely determine transportation decisions at later stages.

This paper presents the first polynomial-time algorithm for this problem (in the length of planning horizon and the number of stages in the supply chain). Although the running time of our algorithm, at $O(LT^8)$, is likely too high for practical purposes, and indeed it is likely that practical instances can be solved by using commercial integer programming software after reformulating and approximating the problem with piecewise-linear cost functions, our focus here is on theoretical advancements, and in particular on introducing the concept of a *basis path*. Traditionally, lot-sizing problems are solved by sequentially iterating over time periods. However, by instead iterating over the basis path, we are able to sufficiently reduce the state space of the problem. We believe that this concept of the basis path has the potential to be applied to other discrete-time optimization problems, particularly those that can be modeled as concave-cost network flow problems. The model considered in this paper is limited to a capacity at the first stage. Our solution approach based on a basis path can be extended to other problems, such as multistage lot-sizing problems where midlevel operations are constrained by capacity (e.g., transportation decisions are constrained, a multilevel extension of the two-stage model considered in Lee et al. 2003). We believe that our basis path approach can be used to solve these and similar problems in polynomial time.

Supplementary Material

Supplemental material to this paper is available at http://dx.doi.org/10.1287/opre.1120.1141 (Appendices S.1–S.3).

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