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Sufficient matrices belong to L

We dedicate this article to the memory of George B. Dantzig and Carlton E. Lemke who continue to inspire researchers everywhere.

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Abstract. In this paper, we establish a significant matrix class inclusion that seems to have been overlooked in the literature of the linear complementarity problem. We show that P_*, the class of sufficient matrices, is a subclass of L. In the course of demonstrating this inclusion, we introduce other new matrix classes that forge interesting new connections between known matrix classes.

1. Introduction

Matrix classes have always played a prominent role in the study of the Linear Complementarity Problem (LCP): Given $M \in \mathcal{R}^{n\times n}$ and $q \in \mathcal{R}^{n\times 1}$, find a vector $x \in \mathcal{R}^n$ such that

$$\begin{aligned} x &\geq 0, \\ q + Mx &\geq 0, \\ x^T(q + Mx) &= 0. \end{aligned}$$

We denote this system by the pair (q, M). Its *feasible set* (the vectors that satisfy the two sets of linear inequalities) is denoted FEA (q, M), whereas its *solution set* (the set of feasible vectors satisfying the third ("complementarity") condition) is denoted SOL (q, M). Abundant coverage of matrix classes in the LCP is available in the monographs [6], [24], and the research articles in the reference list below.

The present study began as an effort to answer a question in the theory of the LCP that seems not to have been posed before: Is the class P_* of sufficient matrices included in the class L? We answer this question in the affirmative. Furthermore, we explore the connections of L with other matrix classes. In so doing, we establish a bridge between copositive matrices and certain subclasses of P_0.

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As originally defined in the literature [7], the class of "sufficient matrices" mentioned above was denoted SU. It was conjectured and partially proved in [17] that SU and the class P_* defined by Kojima, Megiddo, Noma, and Yoshise [19] are actually the same. The latter class equals the union of $\mathrm{P}_*(\kappa)$ where κ is a non-negative number. A real $n \times n$ matrix M belongs to $\mathrm{P}_*(\kappa)$ if and only if

$$(1+4\kappa) \sum_{i \in I_+(x)} x_i(Mx)_i + \sum_{i \in I_-(x)} x_i(Mx)_i \geq 0 \quad \forall x \in \mathcal{R}^n$$

and

$$I_+(x) = \{i : x_i(Mx)_i > 0\} \quad \text{and} \quad I_-(x) = \{i : x_i(Mx)_i < 0\}.$$

The aforementioned conjecture was established conclusively by Väliaho [25]. Hereafter, we adopt the notation P_* for the class of sufficient matrices. We observe that the basic definition of P_* does not suggest a finite test for membership in that class. Nevertheless, by virtue of the equivalence of P_* and SU, such a test is available. See [5].

It is known that P_* contains several other classes of interest in the study of the LCP. Among these are: the positive semidefinite matrices, the matrices whose principal minors are all positive, the matrices having only one principal minor of value zero and all the rest positive, and the adequate matrices. Interestingly, all these subclasses of P_* are known to be subclasses of the class L which was introduced by Eaves in [11] where many of these inclusions were first proved. They contributed directly to our hunch that $\mathrm{P}_* \subset \mathrm{L}$. It should be noted that $\mathrm{L} \subset \mathrm{Q}_0$, meaning that whenever (q, M) is feasible, it is solvable. Indeed, Eaves showed that for $M \in \mathrm{L}$, Lemke's algorithm [20] will process (q, M), that is, find a solution or give evidence that the problem is infeasible.

2. Notation and terminology

In this section we set down a bit of notation and give a few standard definitions to get things started. Others will be given later, as needed. Our aim, for the moment, is to enable the reader to understand the material up to this point.

All non-indexed matrices and vectors are in $\mathcal{R}^{n \times n}$ and $\mathcal{R}^n$, respectively, but we write $(x_1, \ldots, x_n)$ instead of $[x_1 \cdots x_n]^T$. If A is a matrix, $A_{\bullet i}$ denotes the ith column of A. Whenever $\alpha \subseteq \{1, 2, \ldots, n\}$ we denote by $\bar{\alpha}$ the complement of α with respect to $\{1, 2, \ldots, n\}$.

Definition 2.1. *A matrix $M \in \mathrm{E}_0$ if for every $0 \neq x \geq 0$ there exists some i such that $x_i > 0$ and $(Mx)_i \geq 0$. Introduced in [11] and [18], the matrices in this class are said to be* semimonotone. *See [6,* **3.13.18***]*

Definition 2.2. *A matrix $M \in \mathrm{E}_1$ if for every nonzero $x \in \mathrm{SOL}\,(0, M)$ there exists nonnegative diagonal matrices Γ and Ω such that $(\Gamma M + M^T\Omega)x = 0$ and $\Omega x \neq 0$.*

Definition 2.3. $\mathrm{L} = \mathrm{E}_0 \cap \mathrm{E}_1$.

Definition 2.4. *A matrix $M \in$ CSU (is Column Sufficient) if for all x*

$$x_i(Mx)_i \leq 0 \quad (i = 1, 2, \ldots, n) \quad \textit{implies that} \quad x_i(Mx)_i = 0 \quad (i = 1, 2, \ldots, n).$$

Definition 2.5. *A matrix $M \in$ RSU (is Row Sufficient) if $M^T \in$ CSU.*

Definition 2.6. $\mathrm{P}_* = \mathrm{CSU} \cap \mathrm{RSU}$. *The matrices in this class are said to be* sufficient.

Remark. It is known [7, p. 238] that RSU (and hence CSU) is a subclass of P_0, the class of matrices with nonnegative principal minors. This fact makes RSU and CSU subclasses of E_0 as we know from [11].

Definition 2.7. *Given a vector x, we shall denote by $\sigma(x)$ the* support *of x, i.e. $\{i \mid x_i \neq 0\}$.*

For a given matrix $M \in \mathcal{R}^{n\times n}$ and a given partition $\alpha, \bar{\alpha}$ of $\{1, 2, \ldots, n\}$, $C_M(\alpha)$ denotes the $n \times n$ matrix for which

$$[C_M(\alpha)]_{\bullet i} = \begin{cases} -M_{\bullet i} & i \in \alpha \\ I_{\bullet i} & i \in \bar{\alpha} \end{cases} \tag{2.1}$$

With a slight abuse of language, the matrices $C_M(\alpha)$ are called *complementary submatrices* (of $[-M\ \ I]$). The sets

$$pos\, C_M(\alpha) = \{C_M(\alpha)x \mid x \geq 0\}$$

are the *complementary cones*, relative to M. The union over α of these cones is denoted $K(M)$ and called the *complementary range* of M. Clearly, SOL $(q, M) \neq \emptyset$ if and only if $q \in K(M)$.

It is customary to simplify the notation $C_M(\alpha)$ to C_α when the matrix M is understood. Further simplification is achieved (and usually not restrictive) when α is a leading subset of $\{1, 2, \ldots, n\}$, in which case we have

$$C_\alpha = \begin{bmatrix} -M_{\alpha\alpha} & 0 \\ -M_{\bar{\alpha}\alpha} & I \end{bmatrix}. \tag{2.2}$$

This matrix is singular if and only if the principal submatrix $M_{\alpha\alpha}$ of M is singular. When this is the case, the corresponding complementary cone $pos\, C_\alpha$ is said to be *degenerate* and *nondegenerate* otherwise. A *strongly degenerate* complementary cone $pos\, C_\alpha$ is one for which

$$\{x \mid C_M(\alpha)x = 0, \quad 0 \neq x \geq 0\} \neq \emptyset.$$

Strong degeneracy is intimately connected with a criterion for the *boundedness* of the solution set of an LCP, a subject that has been considered in several publications, such as [2], [6], [8], [9], [10] [15] and [21]. Suppose SOL (q, M) is nonempty. Then there exist a positive, but finite, number of index sets α such that $q \in pos\, C_M(\alpha)$. The elements of SOL (q, M) come from the solution sets $X(\alpha)$ of the corresponding systems

$$C_M(\alpha)x = q, \quad x \geq 0.$$

By a well known theorem of linear inequality theory (see for instance [13]) the set $X(\alpha)$ is unbounded if and only if (in our terminology) the complementary cone $pos\, C_\alpha$ is strongly degenerate.

The matrix M belongs to Q_0 when for all q

$$\mathrm{FEA}\,(q, M) \neq \emptyset \Longrightarrow \mathrm{SOL}\,(q, M) \neq \emptyset.$$

It was shown by Eaves [11] that $M \in \mathrm{Q}_0$ if and only if $K(M)$ is convex. This result is also covered in [6, **3.2.1**].

It was mentioned in Section 1, that the class L is a subclass of Q_0. In fact, this is also true of the class RSU as first proved in [7]. It was noted there – and proved in [1] – that since $\mathrm{RSU} \subset \mathrm{P}_0 \cap \mathrm{Q}_0$, Lemke's algorithm [20] will process any LCP (q, M) with a row sufficient matrix M. Furthermore, it was shown in [3] that the Principal Pivoting Method of Cottle and Dantzig also processes this class of linear complementarity problems.

3. Preliminary results

We start by proving a series of lemmas which will be used to show that the class P_* of sufficient matrices is a subclass of L. Since $\mathrm{L} = \mathrm{E}_0 \cap \mathrm{E}_1$, and it is known [7] that $\mathrm{P}_* \subset \mathrm{P}_0 \subset \mathrm{E}_0$, the task boils down to proving $\mathrm{P}_* \subset \mathrm{E}_1$.

Remark. We establish the following three lemmas for a new matrix class even larger than RSU (which, in turn, is even larger than P_*). We believe these easily proved results are of independent interest because they seem to embrace both RSU and C_0, the latter being the class of all *copositive matrices*, i.e., those for which

$$x \geq 0 \Longrightarrow x^T M x \geq 0.$$

It will be helpful to define the polyhedral cone

$$T(M) = \{y \mid y \geq 0,\ y^T M \leq 0\}.$$

(This cone appears in Gowda's paper [14] where copositive star matrices were introduced. See also [6, **3.8.13**].) Note that for *any* square matrix M, we have

$$\mathrm{SOL}\,(0, -M^T) \subseteq \mathrm{FEA}\,(0, -M^T) = T(M).$$

Definition 3.1. *Let* T_* *denote the class of all square matrices* M *such that*

$$T(M) = \mathrm{SOL}\,(0, -M^T).$$

Lemma 3.1. $\mathrm{RSU} \subset \mathrm{T}_*$.

Proof. Let $M \in \mathrm{RSU}$. We need only prove $T(M) \subseteq \mathrm{SOL}\,(0, -M^T)$. For all $y \in T(M)$, we have $y_i(M^T y)_i \leq 0$ $(i = 1, 2, \ldots, n)$. However, $M \in \mathrm{RSU}$, so $M^T \in \mathrm{CSU}$, which implies that $y_i(M^T y)_i = 0$ $(i = 1, 2, \ldots, n)$. Thus we can conclude that $T(M) \subseteq \mathrm{SOL}\,(0, -M^T)$. Hence $M \in \mathrm{T}_*$. □

Remark. The class T_* contains more than just RSU. For example, it contains C_0. Indeed, let M be copositive. Then by the above, $\mathrm{SOL}\,(0, -M^T) \subseteq T(M)$. Now if $y \in T(M)$, then $y^T M y \leq 0$ with $y \geq 0$ so that $y^T M y = 0$; hence $T(M) \subseteq \mathrm{SOL}\,(0, -M^T)$. In short, $\mathrm{C}_0 \subset \mathrm{T}_*$ (strictly).

Lemma 3.2. *Suppose* $M \in \mathrm{T}_*$. *If* $y, z \in \mathrm{SOL}\,(0, -M^T)$, *then* $y_i(M^T z)_i = 0$ $(i = 1, 2, \ldots, n)$ *and* $z_i(M^T y)_i = 0$ $(i = 1, 2, \ldots, n)$.

Proof. Suppose $y, z \in \mathrm{SOL}\,(0, -M^T)$. Since by Definition 3.1, $\mathrm{SOL}\,(0, -M^T)$ is a convex cone, we have $\frac{1}{2}y + \frac{1}{2}z \in \mathrm{SOL}\,(0, -M^T)$. Thus,

$$\begin{aligned} 0 &= (\frac{1}{2}y + \frac{1}{2}z)^T(-M^T)(\frac{1}{2}y + \frac{1}{2}z) \\ &= \frac{1}{4}(y^T(-M^T)y + z^T(-M^T)z - y^T M^T z - z^T M^T y) \\ &= -\frac{1}{4}(y^T M^T z + z^T M^T y). \end{aligned}$$

Noticing that $y \geq 0$, $z \geq 0$, $M^T y \leq 0$, $M^T z \leq 0$, completes the proof. □

Lemma 3.3. *Suppose* $M \in \mathrm{T}_*$ *and* $T(M) \neq \{0\}$. *There exists* β, *a nonempty subset of* $\{1, 2, \ldots, n\}$, *such that for every* $y \in T(M)$, $(M^T y)_\beta = 0$ *and, when* $\bar{\beta}$ *is nonempty,* $y_{\bar{\beta}} = 0$.

Proof. Let $i \in \beta$ if, and only if, there exists $z \in T(M)$ such that $z_i > 0$; that is to say,

$$\beta = \bigcup_{z \in T(M)\setminus\{0\}} \sigma(z).$$

The index set β is nonempty; moreover, since $T(M)$ is a convex cone, β is, in fact, the support of some vector $\hat{z} \in T(M)$. Thus, by Lemma 3.2, if $y \in T(M)$, then $(M^T y)_\beta = 0$, and obviously for nonempty $\bar{\beta}$, $y_{\bar{\beta}} = 0$. □

At this juncture, we verify an assertion which, in essence, was made long ago by Garcia [12].

Proposition 3.1. *If* $M \in \mathcal{R}^{n\times n}$, *then* $M \in \mathrm{E}_1$ *if and only if for every* $z \in \mathrm{SOL}\,(0, M) \setminus \{0\}$, *there exists* y *such that:*

$$y \in T(M) \setminus \{0\} \tag{3.1}$$

$$\sigma(y) \subseteq \sigma(z) \tag{3.2}$$

$$\sigma(M^T y) \subseteq \sigma(Mz) \tag{3.3}$$

Proof. Suppose $M \in \mathrm{E}_1$. If $z \in \mathrm{SOL}\,(0, M)\setminus\{0\}$ there exist nonnegative diagonal matrices Γ and Ω such that $\Omega z \neq 0$ and $(\Gamma M + M^T\Omega)z = 0$. Now if $\mathrm{SOL}\,(0, M) = \{0\}$, there is nothing to prove. Let $z \in \mathrm{SOL}\,(0, M) \setminus \{0\}$. Define $y = \Omega z$. Then y is nonnegative and nonzero. It is then clear that $M^T y \leq 0$, so that (3.1) holds. Furthermore, if $z_i = 0$, then $y_i = 0$. Hence (3.2) holds. Next, if $(Mz)_i = 0$, then $(M^T y)_i = 0$, so (3.3) holds.

Conversely, if $\mathrm{SOL}\,(0, M) = \{0\}$, there is nothing to prove. Hence suppose $z \in \mathrm{SOL}\,(0, M) \setminus \{0\}$ and that there exists a nonzero vector y such that (3.1)–(3.3) hold. By (3.2), it is clear that there exists a nonnegative diagonal matrix Ω such that $y = \Omega z$. By (3.3), there exists a nonnegative diagonal matrix Γ such that $\Gamma M z = -M^T y$. Accordingly, $M \in \mathrm{E}_1$. □

Lemma 3.4. *For a given $M \in \mathcal{R}^{n\times n}$ let $\alpha, \bar{\alpha}$ be a partition of $\{1, 2, \ldots, n\}$. Then there exists a vector y such that*

$$y^T M \leq 0\ ,\ \ 0 \neq y \geq 0$$
$$y_\alpha^T M_{\alpha\alpha} = 0$$
$$y_{\bar{\alpha}} = 0$$

if and only if there is no solution of the system

$$\begin{gathered} C_\alpha x = -Mr + s \\ x \geq 0\ ,\ \ r \geq 0\ ,\ \ s > 0. \end{gathered} \tag{3.4}$$

Proof. It is clear that (3.4) is equivalent to the system

$$\begin{aligned} &C_\alpha x + Mr - s = 0 \\ &s - t = d \text{ (where } d \text{ is an arbitrary positive vector)} \end{aligned} \tag{3.5}$$

$$x \geq 0\ ,\ \ r \geq 0\ ,\ \ s \geq 0\ ,\ \ t \geq 0.$$

By Farkas's lemma (see, e.g., [6, p. 109]), (3.5) has no solution if and only if there exist y and u such that

$$y^T C_\alpha \leq 0, \qquad y^T M \leq 0, \qquad y - u \geq 0, \qquad 0 \neq u \geq 0,$$

which, by setting $y = (y_\alpha, y_{\bar{\alpha}})$ and using (2.1), completes the proof. □

Corollary 3.1. *For a given $M \in \mathcal{R}^{n\times n}$, and a partition $\alpha, \bar{\alpha}$ of $\{1, 2, \ldots, n\}$ there exists y such that*

$$y \in T(M) \setminus \{0\} \tag{3.6}$$
$$\sigma(y) \subseteq \alpha \tag{3.7}$$
$$\sigma(M^T y) \subseteq \bar{\alpha} \tag{3.8}$$

if and only if int pos$[-M \quad I] \cap$ pos $C_\alpha = \emptyset$.

Proof. The assertion follows directly from Lemma 3.4 and the observation that $q \in$ *int pos*$[-M, I]$ if and only if there exist $r \geq 0\ ,\ \ s > 0$ such that $q = -Mr + s$. □

Definition 3.2. *A matrix M belongs to* R_1 *if* SOL (q, M) *is bounded for all $q \in$ int pos $[-M \quad I]$.*

Remark. The class R_1 contains the class R_0. The latter class was introduced in [12] by Garcia who denoted it by $E^*(0)$.The class R_0 is discussed in [6, **3.9.23**] where it is shown that such matrices M are characterized by the boundedness of the solution sets for *all* LCPs (q, M), not just those for which $q \in$ *int pos*$[-M, I]$ as in the case of R_1-matrices.

Taking account of the criterion for unboundedness of the solution set of an LCP (stated above), we have

Lemma 3.5. *$M \in \mathrm{R}_1$ if and only if for every $\alpha \subseteq \{1, 2, \ldots, n\}$*

$$\{0 \neq x \geq 0 \mid C_\alpha x = 0\} \neq \emptyset \Rightarrow int\ pos[-M, I] \cap pos\ C_\alpha = \emptyset. \qquad \square$$

We now return to T_* and give a criterion for such a matrix to belong to the class E_1. This will lead to a criterion for membership in L.

Proposition 3.2. $\mathrm{T}_* \cap \mathrm{R}_1 \subset \mathrm{T}_* \cap \mathrm{E}_1$.

Proof. Suppose $z \in \mathrm{SOL}\,(0, M) \setminus \{0\}$ and let $\sigma_1 = \sigma(z)$, $\sigma_2 = \sigma(Mz)$ and σ_3 be the complement of $\sigma_1 \cup \sigma_2$ with respect to $\{1, 2, \ldots, n\}$. Define $\alpha = \sigma_1 \cup \{i \in \sigma_3 \mid i \in \bar{\beta}\}$ and its complement $\bar{\alpha} = \sigma_2 \cup \{i \in \sigma_3 \mid i \in \beta\}$ where $\beta, \bar{\beta}$ are defined as in Lemma 3.3 and $T(M) \neq \{0\}$. In the event that $T(M) = \{0\}$, we take $\beta = \emptyset$. For simplicity, we assume that α is the leading index set within $\{1,\ 2,\ \ldots,\ n\}$. Let $x = (z_\alpha,\ (Mz)_{\bar{\alpha}})$. Then

$$C_\alpha x = 0\ ,\ 0 \neq x \geq 0, \qquad \text{where } C_\alpha \text{ is defined as in (2.2).} \tag{3.9}$$

Suppose $M \in \mathrm{R}_1$. Then (3.9) and Lemma 3.5 imply that $int\ pos[-M, I] \cap pos\ C_\alpha = \emptyset$, so by Corollary 3.1 there exists y satisfying (3.6)–(3.8).

Suppose $z_i = 0$ for some $i \in \alpha$. Then, $i \in \bar{\beta}$ by the definition of α, so by Lemma 3.3, $y_i = 0$. Noticing that $y_{\bar{\alpha}} = 0$, we conclude that $\sigma(y) \subseteq \sigma(z)$.

Suppose $(Mz)_i = 0$ for some $i \in \bar{\alpha}$. Then $i \in \beta$ by the definition of $\bar{\alpha}$. Thus, by Lemma 3.3, $(M^T y)_i = 0$. Noticing that $(M^T y)_\alpha = 0$, we conclude that $\sigma(M^T y) \subseteq \sigma(Mz)$.

Thus, in view of Proposition 3.1 we have shown that $M \in \mathrm{E}_1$. $\square$

Using Proposition 3.2, and recalling that $\mathrm{L} = \mathrm{E}_0 \cap \mathrm{E}_1$ and $\mathrm{RSU} \subset \mathrm{E}_0 \cap \mathrm{T}_*$, we can immediately state the following corollary which sets up a sufficient condition for a matrix $M \in \mathrm{RSU}$ to be in L.

Corollary 3.2. *Suppose $M \in \mathrm{RSU}$. If $M \in \mathrm{R}_1$, then $M \in \mathrm{L}$.*

4. The main results

We are now in a position to establish our main results.

Theorem 4.1. $\mathrm{CSU} \cap \mathrm{Q}_0 \subset \mathrm{R}_1$.

Proof. Suppose M is a matrix belonging to $\mathrm{CSU} \cap \mathrm{Q}_0$ but not to R_1. Then in view of Lemma 3.5 it must be the case that for some element $\tilde{q} \in int\ pos[-M \quad I]$, we have vectors z and $z^\lambda = \bar{z} + \lambda z$ such that $z \in \mathrm{SOL}\,(0, M) \setminus \{0\}$ and $z^\lambda \in \mathrm{SOL}\,(\tilde{q}, M)$ for all $\lambda \geq 0$. Let

$$\alpha = \sigma(z), \qquad \beta = \sigma(z + \bar{z}) \setminus \alpha, \qquad \gamma = \overline{\alpha \cup \beta}.$$

Next choose $p \in K(M)$ such that

$$\begin{aligned} p_i - \tilde{q}_i &< 0 \quad \text{if } i \in \alpha \\ p_i - \tilde{q}_i &= 0 \quad \text{if } i \in \beta \\ p_i - \tilde{q}_i &> 0 \quad \text{if } i \in \gamma. \end{aligned}$$

Such a vector p is guaranteed to exist since $\tilde{q} \in int\ pos[-M \quad I]$ and $M \in \mathrm{Q}_0$.

Choose any $w \in \mathrm{SOL}\,(p, M)$. The remainder of the argument will show that for sufficiently large λ the vector $z^\lambda - w$ violates the column sufficiency property of M. Note that for all such λ, we have $(z^\lambda - w)_\alpha > 0$. Consequently we have

$$(M(z^\lambda - w))_\alpha = -\tilde{q}_\alpha - (Mw)_\alpha \leq -\tilde{q}_\alpha + p_\alpha < 0.$$

Thus, we have shown

$$(z^\lambda - w)_\alpha (M(z^\lambda - w))_\alpha < 0.$$

Now let $i \in \beta$. Then $\bar{z}_i > 0$ and $z_i = 0$. For such i we also have $p_i = \tilde{q}_i$. For all these i we have

$$\begin{aligned}(z^\lambda - w)_i (M(z^\lambda - w))_i &= (z^\lambda - w)_i ((Mz^\lambda)_i - (Mw)_i + \tilde{q}_i - p_i) \\ &= z_i^\lambda ((Mz^\lambda)_i + \tilde{q}_i) - w_i ((Mz^\lambda)_i + \tilde{q}_i) \\ &\quad + z_i^\lambda (-(Mw)_i - p_i) - w_i (-(Mw)_i - p_i) \\ &\leq 0.\end{aligned}$$

Finally, let $i \in \gamma$. In this case, we must have $z_i = \bar{z}_i = z_i^\lambda = 0$. From this we obtain

$$\begin{aligned}(z^\lambda - w)_i (M(z^\lambda - w))_i &= -w_i ((Mz^\lambda)_i - (Mw)_i) \\ &\leq -w_i ((Mz^\lambda)_i - (Mw)_i + \tilde{q}_i - p_i) \\ &= -w_i (Mz^\lambda + \tilde{q})_i + w_i (Mw + p)_i \\ &= -w_i (Mz^\lambda + \tilde{q})_i \\ &\leq 0.\end{aligned}$$

Thus, we have shown that for sufficiently large λ

$$(z^\lambda - w)_i (M(z^\lambda - w))_i \leq 0 \quad \text{for all } i = 1, \ldots, n;$$

but since $\alpha \neq \emptyset$, it is *not* the case that for sufficiently large λ

$$(z^\lambda - w)_i (M(z^\lambda - w))_i = 0 \quad \text{for all } i = 1, \ldots, n.$$

Since M is column sufficient, this is a contradiction. Hence $M \in \mathrm{R}_1$. □

This brings us to the first of the main results we set out to prove.

Corollary 4.1. *Every sufficient matrix belongs to the class* L.

Proof. Suppose that $M \in \mathrm{P}_*$. Recall that $\mathrm{P}_* = \mathrm{CSU} \cap \mathrm{RSU} \subset \mathrm{RSU} \subset \mathrm{Q}_0$. Accordingly, $\mathrm{P}_* \subset \mathrm{CSU} \cap \mathrm{Q}_0$, so by Theorem 4.1, $M \in \mathrm{R}_1$. Thus, in view of Corollary 3.2, we can conclude that $M \in \mathrm{L}$. □

The reverse of Proposition 3.2—without the restriction that $M \in \mathrm{T}_*$—follows from a result of Gowda and Sznajder [16, Theorem 11]. Here we prove it afresh and thereby avoid developing their conceptual apparatus.

Proposition 4.1. $\mathrm{E}_1 \subset \mathrm{R}_1$.

Proof. Suppose $M \in \mathrm{E}_1$ and let $\alpha, \bar{\alpha}$ be a partition of $\{1, 2, \ldots, n\}$ where for simplicity, α is written as a leading index set within $\{1, 2, \ldots, n\}$. Suppose there exists x such that

$$C_\alpha x = 0, \quad 0 \neq x \geq 0, \qquad \text{where } C_\alpha \text{ is defined as in (2.2).}$$

Let $z \equiv (z_\alpha, z_{\bar{\alpha}}) = (x_\alpha, 0)$, so $Mz = ((Mz)_\alpha, (Mz)_{\bar{\alpha}}) = (0, x_{\bar{\alpha}})$ and $z \in \mathrm{SOL}\,(0, M) \setminus \{0\}$ with $\sigma(z) \subseteq \alpha$ and $\sigma(Mz) \subseteq \bar{\alpha}$. By Proposition 3.1 there exists y satisfying (3.1)–(3.3). Obviously y satisfies (3.6)–(3.8) as well, so by Corollary 3.1, $int\ pos[-M \;\; I] \cap pos\ C_\alpha = \emptyset$, which, in view of Lemma 3.5, implies that $M \in \mathrm{R}_1$. □

Combining Propositions 3.2 and 4.1, and recalling that $\mathrm{L} = \mathrm{E}_0 \cap \mathrm{E}_1$, we get a complete characterization of the class L within $\mathrm{E}_0 \cap \mathrm{T}_*$.

Corollary 4.2. *If* Y *is a subclass of* $\mathrm{E}_0 \cap \mathrm{T}_*$ *then*

$$\mathrm{Y} \cap \mathrm{R}_1 = \mathrm{Y} \cap \mathrm{L}. \qquad \square$$

The significance of this corollary for the classes RSU and C_0 is obvious.

Interestingly, the characterization above can be alternatively expressed in terms of a known (see, for example, [2] and [6, **6.3.14**]) class of matrices which is defined below.[1]

Definition 4.1. *A matrix M belongs to* R_2 *if* $\mathrm{SOL}\,(q, M)$ *is bounded for all $q \in int\ K(M)$.*

It is clear that $\mathrm{R}_1 \subset \mathrm{R}_2$, but more can be said.

Proposition 4.2. *Suppose $M \in \mathrm{E}_0 \cap \mathrm{T}_*$. Then $M \in \mathrm{R}_1$ if and only if $M \in \mathrm{R}_2 \cap \mathrm{Q}_0$.*

Proof. Suppose $M \in \mathrm{R}_2 \cap \mathrm{Q}_0$. Recall that $M \in \mathrm{Q}_0$ means that $K(M) = pos[-M \;\; I]$, which, by definition, implies that $M \in \mathrm{R}_1$.

Suppose $M \in \mathrm{R}_1$. From the assumption $M \in \mathrm{E}_0 \cap \mathrm{T}_*$ and Proposition 3.3, we see that $M \in \mathrm{E}_0 \cap \mathrm{E}_1 = \mathrm{L} \subset \mathrm{Q}_0$. Thus, since $\mathrm{R}_1 \subset \mathrm{R}_2$, we conclude that $M \in \mathrm{R}_2 \cap \mathrm{Q}_0$. □

This, in light of Corollary 4.2, leads to a characterization of the class L within $\mathrm{E}_0 \cap \mathrm{T}_*$ in terms of class R_2.

Corollary 4.3. *If* Y *is a subclass of* $\mathrm{E}_0 \cap \mathrm{T}_*$, *then*

$$\mathrm{Y} \cap \mathrm{R}_2 \cap \mathrm{Q}_0 = \mathrm{Y} \cap \mathrm{L}. \qquad \square$$

Remark. At this time, it is an open question as to whether there exist row sufficient matrices that do not belong to L. Such an example would be needed to demonstrate that restricting RSU to be in R_1 (or, equivalently, in $\mathrm{R}_2 \cap \mathrm{Q}_0$) is required to make it a subclass of L. On the other hand, an example such as

$$M = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$$

[1] The notation R_2 for this class of matrices appears for the first time in this paper.

shows that $P_0 \cap Q_0$ is not a subclass of L. Furthermore, Murthy, Parthasarathy, and Ravindran [23] show that the matrix

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

belongs to $C_0 \cap Q$ but not to R_0. It is elementary to show that it does not belong to E_1 (and therefore to L) either.

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