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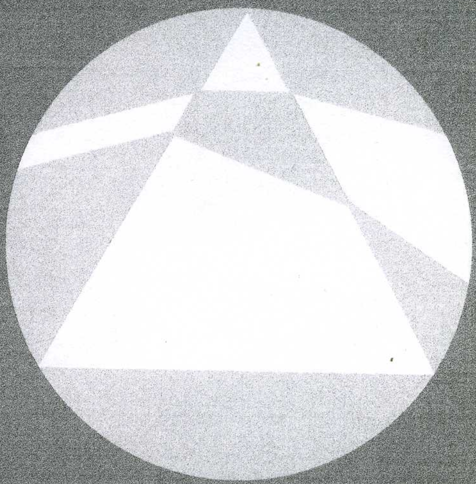

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Fundamental Transformations of Sudoku Grids

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The Sudoku puzzle can be described as follows. Given a 9×9 grid with nine distinct 3×3 subgrids, we say that an assignment of the numerals $1, 2, \ldots, 9$ to the 81 cells of the grid is called a *valid pattern* if every row, column, and subgrid contains all nine numerals (see figure 1 for an example of a valid pattern).

In reference 1 Felgenhauer and Jarvis enumerated the number of valid patterns. In a follow-up article (reference 2), Russell and Jarvis restricted the enumeration to what they called *essentially different patterns*. For that purpose they introduced a list of several transformations of the cells of the grid such as reflections, rotations, and a few others that transform any valid pattern; they considered two patterns to be essentially the same if one can be obtained from the other by a sequence of these transformations. A natural question that can be asked is whether there exist other transformations besides those introduced in reference 2 that should be considered as leaving two patterns essentially the same. In this article we show that any conceivable transformation that leaves a valid pattern essentially the same can be constructed as a finite sequence of the transformations listed in reference 2, thus validating the completeness of the enumeration performed in reference 2 as the precise number of essentially different valid Sudoku patterns.

We denote a given grid by A, where A_{ij} denotes the cell in row i and column j (see figure 2). A particular valid pattern can then be presented by assigning the appropriate numerals to the cells. So, for example, the valid pattern in figure 1 is $A_{11} = 5$, $A_{12} = 2$, and so on.

Note that permuting the numerals (e.g. exchanging 1 and 2) in a valid pattern leaves it essentially the same. In order to eliminate the dependence on particular numeral choices, we say that a partition $S_1, S_2, \ldots, S_9$ of the 81 cells in a grid is *valid* if, for $i = 1, 2, \ldots, 9$,

1. each S_i contains nine of the cells in the grid,
2. every row, column, and subgrid of A contains one, and only one, element of S_i.

5	2	3	8	1	6	7	4	9
7	8	4	5	9	3	1	2	6
6	9	1	4	7	2	8	3	5
2	3	9	1	4	5	6	8	7
4	5	7	2	6	8	9	1	3
1	6	8	9	3	7	2	5	4
3	4	2	7	8	9	5	6	1
9	1	5	6	2	4	3	7	8
8	7	6	3	5	1	4	9	2

Figure 1

A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}	A_{17}	A_{18}	A_{19}
A_{21}	A_{22}	A_{23}	A_{24}	A_{25}	A_{26}	A_{27}	A_{28}	A_{29}
A_{31}	A_{32}	A_{33}	A_{34}	A_{35}	A_{36}	A_{37}	A_{38}	A_{39}
A_{41}	A_{42}	A_{43}	A_{44}	A_{45}	A_{46}	A_{47}	A_{48}	A_{49}
A_{51}	A_{52}	A_{53}	A_{54}	A_{55}	A_{56}	A_{57}	A_{58}	A_{59}
A_{61}	A_{62}	A_{63}	A_{64}	A_{65}	A_{66}	A_{67}	A_{68}	A_{69}
A_{71}	A_{72}	A_{73}	A_{74}	A_{75}	A_{76}	A_{77}	A_{78}	A_{79}
A_{81}	A_{82}	A_{83}	A_{84}	A_{85}	A_{86}	A_{87}	A_{88}	A_{89}
A_{91}	A_{92}	A_{93}	A_{94}	A_{95}	A_{96}	A_{97}	A_{98}	A_{99}

A

Figure 2

A_{63}	A_{43}	A_{53}	A_{73}	A_{93}	A_{83}	A_{33}	A_{23}	A_{13}
A_{61}	A_{41}	A_{51}	A_{71}	A_{91}	A_{81}	A_{31}	A_{21}	A_{11}
A_{62}	A_{42}	A_{52}	A_{72}	A_{92}	A_{82}	A_{32}	A_{22}	A_{12}
A_{68}	A_{48}	A_{58}	A_{78}	A_{98}	A_{88}	A_{38}	A_{28}	A_{18}
A_{67}	A_{47}	A_{57}	A_{77}	A_{97}	A_{87}	A_{37}	A_{27}	A_{17}
A_{69}	A_{49}	A_{59}	A_{79}	A_{99}	A_{89}	A_{39}	A_{29}	A_{19}
A_{64}	A_{44}	A_{54}	A_{74}	A_{94}	A_{84}	A_{34}	A_{24}	A_{14}
A_{65}	A_{45}	A_{55}	A_{75}	A_{95}	A_{85}	A_{35}	A_{25}	A_{15}
A_{66}	A_{46}	A_{56}	A_{76}	A_{96}	A_{86}	A_{36}	A_{26}	A_{16}

$\Phi_1(A)$

A_{11}	A_{15}	A_{14}	A_{13}	A_{12}	A_{16}	A_{19}	A_{18}	A_{17}
A_{28}	A_{23}	A_{26}	A_{27}	A_{24}	A_{21}	A_{29}	A_{22}	A_{25}
A_{33}	A_{36}	A_{32}	A_{38}	A_{34}	A_{39}	A_{37}	A_{31}	A_{35}
A_{47}	A_{46}	A_{42}	A_{45}	A_{48}	A_{44}	A_{43}	A_{41}	A_{49}
A_{58}	A_{51}	A_{55}	A_{54}	A_{57}	A_{59}	A_{52}	A_{53}	A_{56}
A_{68}	A_{69}	A_{67}	A_{61}	A_{64}	A_{63}	A_{66}	A_{62}	A_{65}
A_{77}	A_{78}	A_{72}	A_{73}	A_{76}	A_{79}	A_{74}	A_{75}	A_{71}
A_{86}	A_{84}	A_{83}	A_{81}	A_{87}	A_{88}	A_{82}	A_{89}	A_{85}
A_{94}	A_{98}	A_{91}	A_{95}	A_{92}	A_{96}	A_{97}	A_{93}	A_{99}

$\Phi_2(A)$

Figure 3

So, in figures 1 and 2, the partition of the grid corresponding to the given pattern is

$$S_1 = \{A_{11}, A_{24}, A_{39}, A_{46}, A_{52}, A_{68}, A_{77}, A_{83}, A_{95}\} \quad \text{(the component corresponding to 5)},$$
$$S_2 = \{A_{51}, A_{72}, A_{23}, A_{34}, A_{45}, A_{86}, A_{97}, A_{18}, A_{69}\} \quad \text{(the component corresponding to 4)},$$

and so on. (It should be noted that throughout this article the ordering of elements within a set is arbitrary. For example, $\{x, y, z\}$ is the same as $\{y, x, z\}$.)

Given a grid A, let Φ be a permutation of the cells in A. Thus, Φ is a one-to-one transformation of the cells of A to another 9×9 grid, $\Phi(A)$. That is, each of the cells A_{ij} in A is transformed into a cell in $\Phi(A)$ such that no two cells in A are transformed into the same cell in $\Phi(A)$. Figure 3 depicts two examples of such permutations.

Given a subset S of cells in A, we say that $\Phi(S)$ is the *image* of the cells in S under the transformation Φ. We call Φ a *fundamental transformation* if, for every valid partition $S_1, S_2, \ldots, S_9$ of A, $\Phi(S_1), \Phi(S_2), \ldots, \Phi(S_9)$ is a valid partition of $\Phi(A)$. Note that both

(a)

8	9	7	2	6	5	1	4	3
1	2	4	3	8	9	6	7	5
6	3	5	4	7	1	9	8	2
5	8	1	6	9	7	3	2	4
2	6	9	5	4	3	8	1	7
4	7	3	1	2	8	5	6	9
9	1	2	7	3	6	4	5	8
3	4	6	8	5	2	7	9	1
7	5	8	9	1	4	2	3	6

(b)

8	6	2	7	9	5	3	4	1
7	4	9	6	3	1	5	2	8
5	1	3	8	4	2	9	6	7
3	7	8	9	2	6	1	5	4
1	2	4	5	8	7	6	9	3
6	9	5	4	1	3	8	7	2
4	5	1	2	6	8	7	3	9
2	8	6	3	7	9	4	1	5
9	3	7	1	5	4	2	8	6

Figure 4 (a) Φ_1 applied to the Sudoku in figure 1, (b) Φ_2 applied to the Sudoku in figure 1.

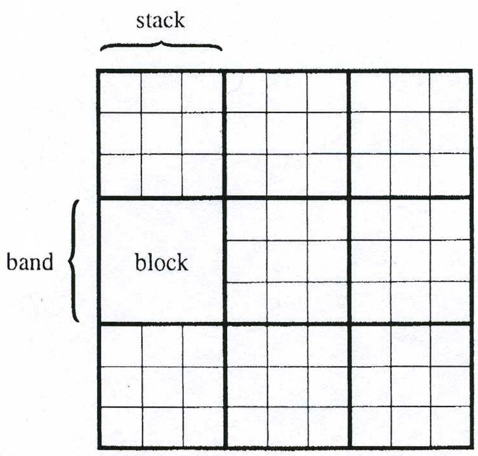

Figure 5

transformations in figure 3 when applied to the particular pattern in figure 1 result in valid partitions (see figure 4). However, as we will later demonstrate, while Φ_1 is a fundamental transformation, Φ_2 is not.

Following the terminology in reference 2, we say that a *block* refers to one of the 3×3 subgrids within the 9×9 grid, a *stack* consists of three blocks in a vertical 9×3 subgrid, and a *band* consists of three blocks in a horizontal 3×9 subgrid (see figure 5).

As discussed in reference 1, there are several well-known simple fundamental transformations that we shall call *elementary* here. Below is the list of the elementary transformations to be considered in this article. (A sixth transformation mentioned in reference 2, rotation, can be obtained by using reflection followed by permutations of stacks and columns.)

1. Permute the three stacks.

2. Permute the three bands.

3. Permute the three columns within a stack.

4. Permute the three rows within a band.

5. Reflection with respect to the main NW-to-SE diagonal (similar to transposing a matrix).

Clearly, any transformation constructed by applying sequentially a series of fundamental transformations is fundamental. We denote the transformation resulting from applying any transformation Φ followed by another transformation ($\bar{\Phi}$, say) as $\bar{\Phi}\Phi$ and the inverse transformation of Φ by Φ^{-1} (that is, $\Phi^{-1}\Phi(A) = A$). Our main result is that any fundamental transformation can be constructed by applying a finite sequence of elementary transformations. Specifically, we prove the following theorem.

Theorem 1 *Let Φ be a fundamental transformation. Then, there exists a sequence of elementary transformations $\Phi_1, \Phi_2, \ldots, \Phi_m$ such that $\Phi = \Phi_m \Phi_{m-1} \cdots \Phi_1$.*

Given a fundamental transformation Φ and a grid A, let $B = \Phi(A)$. We will show that it is always possible to identify a sequence of elementary transformations $\Psi_1, \Psi_2, \ldots, \Psi_m$ such that $\Psi_m \Psi_{m-1} \cdots \Psi_1(B) = A$. Since, as can be easily verified, for any of the five elementary transformations there exists an inverse elementary transformation of the same kind, by applying $\Psi_1^{-1}\Psi_2^{-1}\cdots\Psi_m^{-1}$ on both sides of the preceding equation we get $B = \Psi_1^{-1}\Psi_2^{-1}\cdots\Psi_m^{-1}(A)$, which will prove theorem 1. We shall first prove several preliminary results.

Two cells in a grid A are said to be *dependent within* A if they share a row, a column, or a block within A. Conversely, two cells in a grid A are said to be *independent within* A if they do not share a row, a column, or a block within A. Note that all the cells in any of the components of a valid partition are mutually independent. Thus, given a grid A and a nonfundamental transformation Φ, it means that there exist two independent cells (s and t, say) in A such that $\Phi(s)$ and $\Phi(t)$ are dependent within $\Phi(A)$. Next we prove the converse of the preceding statement.

Lemma 1 *Given a grid A and a transformation Φ, suppose that there exist two independent cells (s and t, say) in A for which $\Phi(s)$ and $\Phi(t)$ are dependent within $\Phi(A)$. Then Φ is nonfundamental.*

Lemma 1 follows directly from the following result.

Lemma 2 *If two cells are independent within A, then there exists a valid partition of A such that these two cells belong to the same component of the partition.*

We can prove lemma 2 by showing that for every valid pattern with two different numerals ('x' and 'y', say) in any two independent cells, we can obtain another valid pattern with numeral 'x' in both cells. Consider the grid in figure 1. Suppose that we want to construct another valid pattern in which the two independent cells A_{11} and A_{54} both are assigned the numeral 5. (In the example, $A_{11} = 5$ and $A_{54} = 2$.) The idea is to exchange the content of cell A_{54} with that of the cell in its block that contains 5 (A_{46} in the grid of figure 1). Now simply switch rows 4 and 5 and then switch columns 4 and 6, placing 5 in A_{54} as desired. The point is that the new pattern is valid since we used only elementary transformations. Thus, we obtain the following theorem.

Theorem 2 *Given a grid A, a transformation Φ is fundamental if and only if, for every pair s, t of independent cells within A, the pair $\Phi(s)$, $\Phi(t)$ is independent within $\Phi(A)$.*

Thus, by theorem 2, Φ_2 (as given in figure 3) is not fundamental since A_{11}, A_{47} is an independent pair of cells within A whereas $\Phi_2(A_{11})$, $\Phi_2(A_{47})$ is a dependent pair of cells within $\Phi_2(A)$ (since both cells appear in the fifth row of $\Phi_2(A)$). Next, we prove a key property of fundamental transformations.

Lemma 3 *Let A be a given grid and Φ a fundamental transformation.*

(a) *Either (i) or (ii) holds.*

- (i) *For every row R (that is, the set of all the cells in the row) in A, $\Phi(R)$ is a row in $\Phi(A)$; for every column C in A, $\Phi(C)$ is a column in $\Phi(A)$.*
- (ii) *For every row R in A, $\Phi(R)$ is a column in $\Phi(A)$; for every column C in A, $\Phi(C)$ is a row in $\Phi(A)$.*

(b) *If B is a block in A, then $\Phi(B)$ is a block in $\Phi(A)$.*

Proof A key observation in the proof is that a set of nine mutually dependent cells in a grid is necessarily a row, a column, or a block. Now, since the number of pairs of independent cells in a grid A is fixed, we have, by theorem 2, that a transformation Φ is fundamental if and only if every dependent pair of cells is transformed to a dependent pair of cells. Consequently, if S is a set of nine mutually dependent cells in a grid A and Φ is a fundamental transformation, then $\Phi(S)$ is necessarily a row, a column, or a block in $\Phi(A)$. Thus, given a row R and a column C in A and since Φ is a fundamental transformation, we have that each of $\Phi(R)$ and $\Phi(C)$ is a row, column, or block in $\Phi(A)$. However, the union of R and C contains 17 cells so the union of $\Phi(R)$ and $\Phi(C)$ also contains 17 cells. But the union of a block and a row (or a column) contains either 15 or 18 cells while the union of two distinct rows (or two distinct columns, or two distinct blocks) contains 18 cells. This proves part (a) of the lemma. Moreover, there are 27 distinct sets of nine mutually dependent sets in A (as well as in $\Phi(A)$), namely the rows, columns, and blocks, which given (a) proves (b).

An immediate consequence of lemma 3 is the following corollary.

Corollary 1 *Given a stack S and a band N in a grid A and Φ a fundamental transformation, the following holds.*

- (i) *If lemma 3(a)(i) holds, then $\Phi(S)$ is a stack in $\Phi(A)$ and $\Phi(N)$ is a band in $\Phi(A)$.*
- (ii) *If lemma 3(a)(ii) holds, then $\Phi(S)$ is a band in $\Phi(A)$ and $\Phi(N)$ is a stack in $\Phi(A)$.*

Finally, we are ready to prove theorem 1. As was stated earlier, we shall prove theorem 1 by showing that we can transform $B = \Phi(A)$ to A by a sequence of elementary transformations. We shall proceed by introducing several steps where each step is designed to transform a particular subgrid of B.

Proof of theorem 1 We split the proof into three steps.

1. According to lemma 3, all the rows in B are transformations of either rows or columns of A. If it is the latter, reflect the entries of B along the main NW-to-SE diagonal to obtain B_1.

2. Given (as established in step 1) that every row in A is a row in B_1, every column in A is a column in B_1, and by corollary 1, we get that for any band N in A, $\Phi(N)$ is a band in B_1, and for any stack S in A, $\Phi(S)$ is a stack in B_1. Now, permute the bands and stacks in B_1 to correspond to the bands and stacks in A to give B_2.

3. Finally within each band (or stack) of B_2, permute the rows (or columns) to correspond to those in A to give B_3.

Step 3 completes the proof, as now $B_3 = A$.

We conclude the article with two remarks.

Remark 1 In a recent paper (reference 3), Herzberg and Murty discussed the question regarding the number of valid patterns in any n-*Sudoku*, where the grid has $n^2 \times n^2$ cells with n^2 $(n \times n)$ blocks and the task is to fill up the grid with numerals $1, 2, \dots, n$ such that each numeral appears once and only once in each row, column, and block. In particular, they listed the elementary transformations as those leading to what they call *equivalent* Sudoku patterns. It can be easily confirmed that all the results presented here can be extended directly to any n-Sudoku, thus establishing that the definition of equivalency in reference 3 is justified as it covers all the transformations that preserve valid patterns.

Remark 2 Note that as a corollary to theorem 1, it is possible to generate all the fundamentally valid patterns from an arbitrary valid pattern by apply the six available permutations to each of the eight elementary transformations involving permutations and the two possibilities for the reflection, leading to $6^8 \times 2$ valid patterns ($6^8 \times 2 \times 9! = 1\,218\,998\,108\,160$ if valid patterns with the numerals permuted are considered). However, this number may exceed the actual number of essentially different valid patterns since it is conceivable that two distinct fundamental transformations, $\Phi_1(A)$ and $\Phi_2(A)$, transform a valid partition A to the same partition (that is, $\Phi_1(A) = \Phi_2(A)$). It is interesting to note that the number above is extremely close (within 0.005%) to 1 218 935 174 261, the actual number of valid patterns as computed in reference 2.

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