Incentive Compatible Mechanisms in the Secretary Problem

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Abstract

Optimal threshold policies for many variants of the secretary problem ignore one fundamental feature of applicants: they are rational human beings. As an example, the famous classical optimal threshold rule will interview all applicants, but will reject the first $\frac{n}{e}$ without due consideration, and thus giving them no incentive to show up to the interview in the first place. We say a hiring mechanism is incentive compatible when it selects all applicants with the same probability. Our contributions are two-fold. First, in sharp contrast to previous works, we show incentive-compatibility can be costly to an employer, depending on her hiring objective function. Second, we show Buchbinder, Jain, and Singh's linear program linking feasible solutions and policies is the dual of some appropriately transformed LP for a Markov Decision Process. As such, this gives us a more orderly approach to studying policies from an applicant's perspective.

Keywords: *optimal stopping, secretary problem, linear programming, incentive compatible.*

1 Motivation

An inherent problem with the secretary problem's optimal threshold policy is that it completely ignores applicants' motives. Imagine yourself spending an entire day interviewing, only to find out later that you are not selected for the position. Worse still, this has nothing to do with your qualifications, but rather is due to the employer's hiring scheme. If you are one of those applicants in earlier slots, you are a part of the *learning* phase and are being used as guinea pigs in this selection process. Would you participate in an interview process knowing that you will not have any chance of getting selected? It is reasonable to assume that this is not something anyone would want, and as such, these earlier applicants have strong incentive to not show up to their interview slots in the first place. And this is disaster from the employer's point of view, as she now may not be able to observe and learn from early applicants as the optimal policy was designed to do.

When slot i has a higher probability of getting selected than slot j, an applicant is said to prefer slot i over slot j. We say a hiring mechanism is incentive compatible when each applicant does not prefer other interviewing slots over his own. Thus two important questions arise: first, does there exist an incentive compatible hiring mechanism? And second, if existence is guaranteed, what is the optimal incentive compatible hiring mechanism? Obviously, selecting applicants randomly with equal probability constitutes an incentive compatible hiring mechanism, and hence the first question has an affirmative answer. The key question then, is how much better than random selection can an employer do?

Buchbinder, Jain, and Singh [?] (henceforth will be referred to as BJS) asked and gave the answer for the second question. There, they find that an optimal incentive compatible hiring mechanism selects the best applicant with probability $1 - \frac{1}{\sqrt{2}} \approx 0.29$, as compared to the more well known number $\frac{1}{e} \approx 0.368$. Also observe that random selection hires the best applicant with probability $\frac{1}{n} \to 0$. As such, their incentive compatible hiring mechanism does relatively well against the traditional optimal threshold hiring mechanism, and is a significant improvement over the trivial random selection mechanism. Their approach consists of formulating the secretary problem as a linear program, and adding constraints $p_i^\pi = p_j^\pi$ for all $i \neq j$, which stipulates the mechanism must select all slots with the same probability.

In this paper, we will show how to obtain linear programs derived in [?] as duals of some appropriately transformed linear program for a Markov Decision Process. Furthermore, we will explore other versions of the secretary problem in the same manner as [?], and show many conclusions which are vastly different from what BJS obtained in the traditional setting. One such result is that incentive compatible hiring mechanisms can be **costly** for the employer.

2 Introducing the Rank-Based Secretary Problem

An employer with the classic objective of hiring the best overall applicant is being picky in an extreme way. She is never satisfied with anyone but the best, and may come away from the hiring process empty-handed. This may be okay in certain hypothetical situations (e.g. choosing the best surgeon to perform on one's heart), but is not necessary in many others (e.g. choosing a boy to deliver newspapers).

In the rank-based secretary problem, there are n applicants who apply for one available job. If we are allowed to observe them all, we would be able to rank them individually, from best (rank 1) to worst (rank n). Suppose we assign these applicants to interview slots in a random order, and when the ith applicant is interviewed, we can only observe his rank relative to those who came in before he did. We must make our hiring decision online: either accept him and end the job search, or reject him and continue with the job search. Our goal in this rank-based setting is to find a hiring strategy which minimizes the expected rank of the hiree.

Lindley [?] was the first to consider this version of the secretary problem. He was able to derive a recurrence equation characterizing the optimal threshold policy, which is of the form:

while interviewing for the rth applicant, stop and accept if her apparent rank $s \leq s^*(r)$, continue if $s > s^*(r)$.

The recurrence equation that needs to be solved to get $s^*(r)$ is fairly complex, and it was Chow et. al. [?] who successfully showed the optimal stopping policy chooses an applicant with expected rank $\prod_{j=1}^{\infty} \left(\frac{j+2}{j}\right)^{1/(j+1)} = 3.8695$ as $n \to \infty$.

In the rank-based secretary problem, we seek to minimize the expected rank of the selected applicant. Observe that this objective is **equivalent** to **maximizing the expected utility** of the selected applicant, where the utility for hiring an ith-rank applicant is $(n-i)$. Let r_n^* denote the optimal expected rank of the hiree when there are n applicants, and u_n^* the optimal expected utility of the hiree when there are n applicants. It is evident that the optimal expected rank $r_n^* = n - u_n^*$. We choose to work with this alternative problem of maximizing the expected utility from now on.

2.1 LP Formulation For The Utility-Based Secretary Problem

Our first objective is to derive the linear programming formulation for the incentive compatible utility-based secretary problem. We will give two different approaches to this result. The first uses known results in the theory of Markov Decision Process and linear programming, and is shown below. The second approach is similar in spirit to that taken by [?], and has an algorithmic flavor to it. It will be expounded in the **Appendix**.

Proposition 2.1. *The following linear program is a formulation of the utility-based secretary problem.*

$$\begin{array}{lll} max & \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{n(i-j+1)-j}{i(i+1)} q_i^j & \\ s.t. & q_i^j \leq 1 - \sum_{l<i} p_l & 1 \leq j \leq i \leq n \\ & p_i = \frac{1}{i} \sum_{k=1}^{i} q_i^k & 1 \leq i \leq n \\ & p_i, q_i^j \geq 0 & \end{array}$$

Proof. Consider a Markov Decision Process where the underlying state (i, j) denotes the employer is interviewing the ith applicant given he is jth best so far. The employer has two possible actions, **stop** or **continue**. Let $V_{i,j}$ denotes the optimal value at state (i, j), then we must have the following (see [?]):

$$V_{i,j} = \max\left\{n - \frac{n+1}{i+1}j,\ \frac{1}{i+1}\sum_{l=1}^{i+1} V_{i+1,l}\right\}$$

The above can be formulated as the following linear program (D'):

$$\begin{array}{llcll} \min & V_{1,1} & & & \\ \text{s.t.} & V_{i,j} & \geq & n - \frac{n+1}{i+1}j & 1 \leq j \leq i \leq n \\ & V_{i,j} & \geq & \frac{1}{i+1}\sum\limits_{l=1}^{i+1} V_{i+1,l} & 1 \leq j \leq i \leq n \end{array}$$

Consider the transformation $i \cdot x_{i,j} = V_{i,j} - \frac{1}{i+1}\sum\limits_{l=1}^{i+1} V_{i+1,l}$ for all $1 \leq j \leq i \leq n$. By induction, we can show that

$$V_{i,j} = i \cdot x_{i,j} + \sum_{l=i+1}^{n}\sum_{j=1}^{l} x_{l,j}$$

Let us denote $iy_i = -\sum\limits_{l=i+1}^{n}\sum\limits_{j=1}^{l} x_{l,j}$, then ($D'$) can be re-written as:

$$\begin{array}{lccll} \min & \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{i} x_{i,j} & & & \\ \text{s.t.} & x_{i,j} - y_i & \geq & \frac{n(i-j+1)-j}{i(i+1)} & 1 \leq j \leq i \leq n \\ & \sum\limits_{l=i+1}^{n}\sum\limits_{j=1}^{l} x_{l,j} + iy_i & = & 0 & 1 \leq i \leq n \\ & x_{i,j} \geq 0,\ y_i \text{ free} & & & 1 \leq j \leq i \leq n \end{array}$$

But observe that this newly transformed linear program is the dual of the desired LP by letting q_i^j's to be dual variables corresponding to inequality constraints, and p_i's to be dual variables corresponding to equality constraints.

□

We note the necessity of introducing q_i^j's into the model, as the dynamic program requires knowing an applicant's relative rank at each stage to determine an optimal policy. In this way, different probability variables are needed for different problems. In particular, they must be dependent on the decision maker's objective function and constraints. We choose to delay elaborating on this point until the next section, where another example will be exhibited.

We also note that this proof does not allow us to directly interpret p_i's and q_i^j's as probabilities. To give proper justifications, we need to use the type of proof as outlined in the **Appendix**.

From our previous claim and its discussion, we can now define the linear program for the incentive compatible utility-based secretary problem.

Proposition 2.2. *The following linear program (P) is a formulation of the incentive compatible utility-based secretary problem.*

$$\begin{array}{lll} \mathit{max} & \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{i} \frac{n(i-j+1)-j}{i(i+1)} q_i^j & \\ \mathit{s.t.} & q_i^j \leq 1 - \sum\limits_{l<i} p_l & 1 \leq j \leq i \leq n \\ & p_i = \frac{1}{i}\sum\limits_{k=1}^{i} q_i^k & 1 \leq i \leq n \\ & p_i = p & \\ & q_i^j \geq 0 & \end{array}$$

Here, p_i denotes the probability a policy selects the ith applicant, and q_i^j denotes the probability a policy selects the ith applicant given he is the jth best so far.

2.2 Modeling The Secretary Problem With Backward Solicitation

It should be observed that the method presented earlier can be employed to derive BJS-style linear programs for other secretary problems. The main task in the process is finding an appropriate transformation of variables, which varies from problem to problem. In those cases when employing a BJS approach proves difficult, the technique above can still be used to gain insight to the problem at hand. In this section, we will illustrate the technique for a version of the secretary problem which allows backward solicitation. This variant is central to our later works in modeling a game-theoretic approach to the secretary problem.

Consider the classical secretary objective of maximizing the probability for choosing the best overall applicant. We assume throughout this section that the employer has the option to recall any previous applicant, and each will accept the employer's offer with probability $\alpha(s,i)$. Here, s is the current round of interview, and i is the round when the employer first interviewed the relatively best applicant. As an example, suppose the employer is currently interviewing the 5th applicant, and the applicant in the 3rd round is the best so far, then $s=5$, and $i=3$. If the employer decides to offer the job to the 3rd applicant, he will accept with probability $\alpha(5,3)$. If the applicant does not accept, the employer must move on to interview the next one. Here, we do allow for multiple offers to the same applicant in different rounds (as oppose to [**?**]). That is, if an applicant rejects an offer this round, the employer may still offer that same applicant in the next round, albeit the probability of acceptance may be different (depending on the structure of $\alpha(\cdot,\cdot)$).

2.2.1 Dynamic Programming Formulation

Suppose we are at the position (s,i). That is, the employer is interviewing the sth applicant, and the relatively best applicant is at position i. As in [**?**], let $\pi_f(s,i)$ be the probability of the employer hiring the overall best applicant if she decides to interview the next applicant without solicitating the then current relatively best applicant. Also, let $\pi_b(s,i)$ be the probability of hiring the overall best applicant if she decides to hire the then current relatively best applicant. Let $\pi(s,i)$ be the probability of hiring the overall best applicant. Then we must have the following:

$$\begin{aligned}
\pi_f(s,i) &= \tfrac{1}{s+1}\pi(s+1,s+1)+\tfrac{s}{s+1}\pi(s+1,i)\\
\pi_b(s,i) &= \tfrac{s}{n}\cdot\alpha(s,i)+\pi_f(s,i)\cdot(1-\alpha(s,i))\\
\pi(s,i) &= \max\{\pi_f(s,i),\pi_b(s,i)\}\\
\pi(n,i) &= \alpha(n,i)
\end{aligned}$$

Solving the above dynamic problem will yield the optimal policy that the employer should follow. Instead, we will leave this for the interested readers, and pursue the problem using the linear programming approach.

2.2.2 Linear Programming Formulation

A BJS-style proof can be used to obtain the linear programming formulation, but it is long and may take more efforts than needed. It is presented in the **Appendix**. We show here the approach derived directly from the dynamic program above.

Proposition 2.3. *The secretary problem with backward solicitation can be solved using the linear program below.*

$$\begin{aligned}
&\mathit{max} && \tfrac{1}{n}\sum_{s=1}^{n} s\cdot p_s\\
&\mathit{s.t} && s\cdot p_{s,i}\le\alpha(s,i)\cdot\left(1-\sum_{l<i}p_l-\sum_{l\ge i}^{s-1} l\cdot p_{l,i}\right) && \forall\ 1\le i\le s\le n\\
& && p_s=\sum_{i=1}^{s}p_{s,i} && \forall\ 1\le s\le n\\
& && p_{s,i}\ge 0,\ \ p_s\ \mathit{free}
\end{aligned}$$

Proof. From the dynamic program, we can readily form the following equivalent linear program.

$$\begin{array}{lrcl} \min & \pi(1,1) & & \\ \text{s.t.} & \pi(s,i) & \geq & \frac{1}{s+1}\pi(s+1,s+1) + \frac{s}{s+1}\pi(s+1,i) \\ & \pi(s,i) & \geq & \frac{s}{n}\alpha(s,i) + \left(\frac{1}{s+1}\pi(s+1,s+1) + \frac{s}{s+1}\pi(s+1,i)\right)(1-\alpha(s,i)) \end{array}$$

Next, use the transformation $s \cdot x_{s,i} = \pi(s,i) - \frac{1}{s+1}\pi(s+1,s+1) - \frac{s}{s+1}\pi(s+1,i)$ to obtain the relationship

$$\pi(s,i) = s\sum_{k=s}^{n} x_{k,i} + \sum_{k=s+1}^{n}\sum_{j=k}^{n} x_{j,k}$$

The linear program above can then be transformed into

$$\begin{array}{llcl} \text{minimize} & \sum_{k=1}^{n}\sum_{j=k}^{n} x_{j,k} & & \\ \text{subject to} & x_{s,i} & \geq 0 & \forall\, 1 \leq i \leq s \leq n \\ & \frac{1}{\alpha(s,i)} s \cdot x_{s,i} + s\sum_{k>s}^{n} x_{k,i} + \sum_{k>s}^{n}\sum_{j=k}^{n} x_{j,k} & \geq \frac{s}{n} & \forall\, 1 \leq i \leq s \leq n \end{array}$$

Letting $p_{s,i}$ to be dual variables of the second set of constraints (which corresponds to the **stopping** action on state (s,i)), and form the dual linear program. A straightforward simplification yields what we claimed. □

2.3 General Strategy For Formulations

In general, if we have two possible actions to take, **continue** or **stop**, we should first derive a linear program directly from the set of dynamic programming constraints, then form a BJS-style linear program through the use of a variable transformation for the **continue** constraints. This would make the right hand side of these constraints to be 0, and thus guarantee non-negativity for the transformed variables. As such, when we form the dual linear program, only variables corresponding to the **stop** action would remain in place. These are the p's and q's in Buchbinder et. al.'s and our works. This technique, however, may be difficult to apply in certain cases where the dynamic programming formulation has intricate relationships between variables. As an example, the reader is invited to try formulating a BJS-style linear program for another version of the secretary problem with backward solicitation, where once an applicant has rejected an offer, he cannot be solicited again in future rounds (see [**?**]).

3 An Incentive Compatible Hiring Mechanism Can Be Costly!

Since each feasible solution to the linear program corresponds to a feasible policy, the problem at hand now reduces to optimizing the incentive compatible utility-based LP.

Proposition 3.1. *Define $P(p)$ as the linear program (P) with a fixed $p \in [0, \frac{1}{n}]$. The following feasible solution is optimal for $(P(p))$:*

- $1 \leq i \leq \lfloor \frac{1}{2p} + \frac{1}{2} \rfloor$*: then* $q_i^1 = ip$, $q_i^j = 0$ *for* $j \neq 1$.
- $\lfloor \frac{1}{2p} + \frac{1}{2} \rfloor + 1 \leq i \leq n$*: then* $q_i^1 = \ldots = q_i^k = 1-(i-1)p$, $q_i^{k+1} = ip - k\,(1-(i-1)p)$, *and* $q_i^j = 0$ *for* $k+1 < j \leq n$. *Here,* $k = \lfloor \frac{ip}{1-(i-1)p} \rfloor$.

Proof. First, for a given i, observe that $\frac{n(i-j+1)-j}{i(i+1)}$ is decreasing in j. As $\sum_{j=1}^{i} q_i^j = ip$, we should shift as much as possible into smallest j's. Since $q_i^j \leq 1-(i-1)\cdot p$, the RHS serves as an upper ceiling for each q_i^j.

We consider two scenarios:

- $ip \leq 1-(i-1)p$: then because of the observation above, we should shift everything into q_i^1, so that $q_i^1 = ip$. Note that $ip \leq 1-(i-1)p \iff i \leq \frac{1}{2p} + \frac{1}{2}$.

- $ip > 1-(i-1)p$: this means we can only shift a maximum of $1-(i-1)p$ into each q_i^j. The maximum number of j's that we can shift $(1-(i-1)p)$ into is $k = \lfloor \frac{ip}{1-(i-1)p} \rfloor$. Whatever that is left over, i.e. $ip - k(1-(i-1)p)$, should be shifted to q_i^{k+1}.

$\square$

Proposition 3.2. *Let $u_n^*(p)$ denote the optimal objective value to the problem $P(p)$. Then $u_n^*(p) \leq n - \frac{n+1}{n} \sum_{i=1}^{n} \frac{1}{i+1}$ for all $p \in [0, \frac{1}{n}]$.*

Proof. Consider $p \in [0, \frac{1}{n}]$, then:

$$\begin{aligned} u_n^*(p) &\leq \sum_{i=1}^{n} \frac{n(i-1+1)-1}{i(i+1)} i \cdot p \\ &= \sum_{i=1}^{n} \frac{ni-1}{i+1} p \\ &= np \sum_{i=1}^{n} \left(1 - \frac{1}{i+1}\right) - p \sum_{i=1}^{n} \frac{1}{i+1} \\ &= n^2 p - (n+1)p \sum_{i=1}^{n} \frac{1}{i+1} \\ &\leq n - \frac{n+1}{n} \sum_{i=1}^{n} \frac{1}{i+1} \end{aligned}$$

Justification for the two inequalities are as follows:

- Note that for $1 \leq i \leq \lfloor \frac{1}{2p} + \frac{1}{2} \rfloor$, we have $q_i^1 = ip$ and $q_i^j = 0$ for $j \neq 1$, per **Proposition 3.1**. When i is outside this range, we can shift all of the weight ip to q_i^1, and let $q_i^j = 0$ for other j's. Clearly this new solution is infeasible (it violates the condition $q_i^j \leq 1-(i-1)p$), but it forms an upper bound to the objective function value. Hence the first inequality follows.

- Observe that $n^2 p - (n+1)p \sum_{i=1}^{n} \frac{1}{i+1}$ is increasing in p. Since $p \in [0, \frac{1}{n}]$, the above is maximized at $p = \frac{1}{n}$. Hence the second inequality follows.

$\square$

With the above proposition, we are now in position to show the optimal expected rank of a hired applicant is $\Omega(\log(n))$.

Proposition 3.3. *The optimal expected rank in the envy-free rank-based secretary problem is $\Omega(\log(n))$.*

Proof. Per our observation earlier, $r_n^* = n - u_n^*$. By **Proposition 3.2**, we then have:

$$r_n^* \geq n - \left(n - \frac{n+1}{n} \sum_{i=1}^{n} \frac{1}{i+1}\right) = \frac{n+1}{n} \sum_{i=1}^{n} \frac{1}{i+1} = \Omega(\log(n))$$

$\square$

Recall that in the classical setting with objective of maximizing the probability of selecting the best applicant, introducing incentive compatibility decreases the optimal value from $\frac{1}{e} \approx 0.368$ to $(1-1/\sqrt{2}) \approx 0.293$. When we change the objective to minimizing the expected rank of the hiree, introducing incentive compatibility increases the optimal value from ≈ 3.870 to $\Omega(\log(n))$. As such, incentive compatibility can be costly to the employer depending on her hiring objective.

We next try to form a $\log(n)$ upper bound for the rank-based incentive compatible secretary problem. Similar to the previous case, we start out with a proposition for the utility-based problem.

Proposition 3.4. *Let u_n^* be the optimal value for the linear program (P). Then we must have $u_n^* \in \Omega(n-\log(n))$.*

Proof. Consider the feasible solution in **Proposition 3.1**, take $p = \frac{1}{n}$, and modify q_i^{k+1} to be equal to 0, keeping all other values to be the same. It follows that u_n^* is at least as large as the objective value evaluated at this feasible solution:

$$
\begin{aligned}
u_n^* &\geq \sum_{i=1}^{\lfloor \frac{n}{2}+\frac{1}{2} \rfloor} \tfrac{ni-1}{i(i+1)} \cdot \tfrac{i}{n} + \sum_{i=\lfloor \frac{n}{2}+\frac{1}{2} \rfloor+1}^{n} \sum_{j=1}^{\lfloor \frac{i}{n-i+1} \rfloor} \tfrac{n(i+1)-(n+1)j}{i(i+1)} \left(1 - \tfrac{i-1}{n}\right) \\
&= \sum_{i=1}^{\lfloor \frac{n}{2}+\frac{1}{2} \rfloor} \left(1 - \tfrac{1}{i+1} - \tfrac{1}{(i+1)n}\right) + \sum_{i=\lfloor \frac{n}{2}+\frac{1}{2} \rfloor+1}^{n} \sum_{j=1}^{\lfloor \frac{i}{n-1+1} \rfloor} \left(\tfrac{n-i+1}{i} - \tfrac{n+1}{n} \tfrac{n-i+1}{i(i+1)} \cdot j\right) \\
&\geq \sum_{i=1}^{\lfloor \frac{n}{2}+\frac{1}{2} \rfloor} \left(1 - \tfrac{n+1}{n} \cdot \tfrac{1}{i+1}\right) + \sum_{i=\lfloor \frac{n}{2}+\frac{1}{2} \rfloor+1}^{n} \left(\sum_{j=1}^{\lfloor \frac{i}{n-i+1} \rfloor} \tfrac{n-i+1}{i} - \tfrac{n+1}{n} \sum_{j=1}^{\frac{i}{n-i+1}} \tfrac{n-i+1}{i(i+1)} \cdot j \right) \\
&\geq \sum_{i=1}^{\lfloor \frac{n}{2}+\frac{1}{2} \rfloor} \left(1 - \tfrac{n+1}{n(i+1)}\right) + \sum_{i=\lfloor \frac{n}{2}+\frac{1}{2} \rfloor+1}^{n} \left(1 - 1 + (\tfrac{i}{n-i+1} - 1) \cdot \tfrac{n-i+1}{i} - \tfrac{n+1}{n} \cdot \tfrac{n+1}{2(i+1)(n-i+1)}\right) \\
&\approx n - \tfrac{n+1}{n} \log \tfrac{n}{2} - \log \tfrac{n}{2} - \tfrac{n+1}{n} \cdot \tfrac{n+1}{2(n+2)} \log n
\end{aligned}
$$

Here, the first inequality is simply a comparison between the optimal value and the value of our chosen feasible solution. The second inequality holds due to subtraction of a larger term. The third inequality results from adding and subtracting 1, and the fact that $\lfloor \frac{i}{n-i+1} \rfloor \geq \frac{i}{n-i+1} - 1$. This shows the lower bound is $\Omega(n - \log(n))$. ☐

With these results in hand, we can now conclude the optimal expected rank grows in the order of $\log n$.

Theorem 3.5. *The optimal expected rank $r_n^* = \mathcal{O}(\log n)$, and hence is also in $\Theta(\log n)$.*

Proof. By the above proposition, we know $r_n^* = \mathcal{O}(\log n)$. Due to **Proposition 3.3**, we also know that $r_n^* = \Omega(\log n)$. As such, we conclude that $r_n^* = \Theta(\log n)$. ☐

4 We Always Hire!

In the classical setting of the secretary problem, it is a surprising (and nonobvious) fact that the best incentive compatible policy may not pick anyone at all. And even if it does select someone, that person may not be the best so far at the time of selection. The first of these problems goes away in the rank-based setting, where an optimal incentive compatible hiring policy is guaranteed to always select someone. With regard to the second problem, it is not relevant in the rank-based setting, as our objective is no longer hiring the best overall, but to achieve the minimum expected rank. We state and prove the *Always Hired* property in the following lemma.

Lemma 4.1. *The optimal incentive compatible policy for the utility-based (equivalently rank-based) secretary problem always select someone.*

Proof. (Sketch) It is enough to show for the optimal policy, we must have $p = p_i = \frac{1}{n}$ for all $1 \leq i \leq n$. Equivalently, we take the optimal objective value (using the solution constructed in **Proposition 3.1**) and show it is increasing in p. For the ease of computation, we shall ignore integrality and obtain this objective value below:

$$
\begin{aligned}
u_n^*(p) &= \sum_{i=1}^{1/(2p)} \tfrac{ni-1}{i+1}p + \sum_{i=1/(2p)+1}^{n} \sum_{j=1}^{ip/(1-(i-1)p)} \tfrac{n(i-j+1)-j}{i(i+1)}(1-(i-1)p) \\
&= \sum_{i=1}^{1/(2p)} \tfrac{ni-1}{i+1}p + \sum_{i=1/(2p)+1}^{n} \left(\sum_{j=1}^{ip/(1-(i-1)p)} \tfrac{n(i+1)}{i(i+1)}(1-(i-1)p) - \sum_{j=1}^{ip/(1-(i-1)p)} \tfrac{(n+1)j}{i(i+1)}(1-(i-1)p) \right) \\
&= \sum_{i=1}^{1/(2p)} \tfrac{ni-1}{i+1}p + \sum_{i=1/(2p)+1}^{n} \left(np - \tfrac{n+1}{2i(i+1)}(1-(i-1)p)\tfrac{ip}{1-(i-1)p}\left(\tfrac{ip}{1-(i-1)p}+1\right)\right) \\
&= \tfrac{n}{2} - p(n+1)\sum_{i=1}^{1/(2p)} \tfrac{1}{i+1} + \sum_{i=1/2p+1}^{n} np - \sum_{i=1/2p+1}^{n} \tfrac{p(n+1)}{2(i+1)}\tfrac{ip}{1-(i-1)p} + \sum_{i=1/2p+1}^{n} \tfrac{p(n+1)}{2(i+1)} \\
&= n^2p - p(n+1)\sum_{i=1}^{1/(2p)} \tfrac{1}{i+1} - \tfrac{p(n+1)}{2}\sum_{i=1/2p+1}^{n} \tfrac{1}{i+1}\tfrac{1+p}{1-(i-1)p} \\
&= n^2p - p(n+1)\sum_{i=1}^{1/(2p)} \tfrac{1}{i+1} - \tfrac{p(1+p)(n+1)}{2}\sum_{i=1/2p+1}^{n} \left(\tfrac{1}{1+2p}\tfrac{1}{i+1} + \tfrac{p}{1+2p}\tfrac{1}{1-(i-1)p}\right)
\end{aligned}
$$

Next, observe that the coefficient of the second term is of the order $n\log n$, and the last term's is also at most the order of $n\log n$. As such, the expression is dominated by the coefficient of the first term n^2, and hence shows that $u_n^*(p)$ is increasing in p.

□

It should also be noted that this proof can be used to show $r_n^* \in \Theta(\log n)$, a result that we obtained earlier.

5 Incentive Compatibility In Generalized Utility-Based Problems

In a generalized utility-based secretary problem, the employer will derive a utility of $f(s)$ units for hiring the sth best overall applicant. Modeling this problem as a dynamic program or a linear program is similar to what we have shown previously. If we modify the classical secretary problem, and assign $f(1) = n-1$, and $f(s) = 0$ for all other $s \neq 1$, [?] showed that the optimal incentive compatible policy selects a candidate with expected utility of $\left(1 - \frac{1}{\sqrt{2}}\right)(n-1)$. In this section, we would like to know whether other variations of the utility-based secretary problem will improve upon this, perhaps up to a constant difference from n, the number of applicants. From the previous section, we know the incentive compatible rank-based secretary problem optimally selects an applicant with expected utility in $\Theta(n - \log(n))$. What if the employer has a utility function somewhere in between? In other words, what if $f(s) = n-s$ for $s = 1,\ldots,K$, and $f(s) = 0$ for $s \geq K+1$, with K dependent or independent of n? This section focuses on the asymptotic behavior of such a group of utility function. We now focus on a special class of the utility-based secretary problem, namely that of $f(s) = n-s$ for $s = 1,\ldots,K$, and $f(s) = 0$ for $s = K+1,\ldots,n$. Here, K is assumed to be a fixed number, independent of n.

Proposition 5.1. *For fixed n, $1 \leq i \leq n$, and $1 \leq K \leq n$, then*

$$f_{n,i}^K(j) = \sum_{s=j}^{K}(n-s)\frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}}$$

is a decreasing function in j.

Proof. We wish to show the difference $f_{n,i}^K(j) - f_{n,i}^K(j+1) > 0$. To do this, simply compare the term $(n-j')\frac{\binom{i-1}{j-1}\binom{n-i}{j'-j}}{\binom{n-1}{j'-1}}$ in $f_{n,i}^K(j)$, against the term $(n-j')\frac{\binom{i-1}{(j+1)-1}\binom{n-i}{j'-(j+1)}}{\binom{n-1}{j'-1}}$ in $f_{n,i}^K(j+1)$ for all $j' \geq j+1$. The comparison simplifies to:

$$(n-j')\binom{i-1}{j-1}\binom{n-i}{j'-j} > (n-j')\binom{i-1}{(j+1)-1}\binom{n-i}{j'-(j+1)} \quad \Longleftrightarrow \quad \frac{n-i-(j'-j)+1}{j'-j} > \frac{i-j}{j}$$

The left hand side of the inequality is a decreasing function in j' for $2 \leq j' \leq n$. As such, the partial sum (in terms of K) of $f_{n,i}^K(j) - f_{n,i}^K(j+1)$ is unimodal as K increases (i.e. this difference may first increase with K up to some appropriate K^*, then starts decreasing as K goes beyond this K^*). When $K = j$, clearly this difference is positive. Furthermore, when $K = n$, this difference also stays positive due to the closed-form formula of our utility-based objective function. As such, we can conclude the difference must remain positive for all K in between, i.e. $f_{n,i}^K(j)$ is decreasing in j.

□

Corollary 5.2. *Assign utility $(n-i)$ to the ith best overall applicant for $1 \leq i \leq K$, and weight 0 for all others. For any fixed K, the utility secretary problem has optimal value at most a constant factor of n (depending on K), where n is the number of applicants in the problem.*

Proof. First, observe that the corresponding linear program for this version of the incentive compatible secretary problem is:

$$\begin{array}{lll}
\max & \frac{1}{n}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{i}\sum\limits_{s=j}^{K}(n-s)\frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}}q_i^j & \\
\text{s.t.} & q_i^j \leq 1 - \sum\limits_{l<i} p_l & 1 \leq j \leq i \leq n \\
& p_i = \sum\limits_{j=1}^{i} q_i^j & 1 \leq i \leq n \\
& p_i = p_j = p & 1 \leq i \neq j \leq n \\
& p_i, q_i^j \geq 0 & 1 \leq j \leq i \leq n
\end{array}$$

Because for a fixed i we have $\sum\limits_{s=j}^{K}(n-s)\frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}}$ is decreasing in j (per the preceding proposition), it follows that we should shift as much into q_i^j as possible, before moving on to q_i^{j+1}. The proof then becomes similar to that of **Proposition 3.1**: that for $1 \leq i \leq n$, we can shift all of ip into q_i^1, i.e. $q_i^1 = ip$ and $q_i^j = 0$ for $j \neq 1$ to form an *infeasible* solution. The optimal objective value of this incentive compatible secretary problem then has the following upper bound:

$$\frac{1}{n}\sum_{i=1}^{n}\sum_{s=1}^{K}(n-s)\frac{\binom{i-1}{1-1}\binom{n-i}{s-1}}{\binom{n-1}{s-1}} \cdot ip$$

Next, when n is large, and k is much smaller than n, we have the following approximation asymptotics: $\binom{n}{k} \sim \frac{\left(\frac{2n}{k}-1\right)^k}{\sqrt{2\pi k}}$. Apply this to the upper bound of the optimal objective value above and we obtain:

$$\begin{array}{rcl}
\frac{1}{n}\sum\limits_{i=1}^{n}\sum\limits_{s=1}^{K}(n-s)\frac{\binom{n-i}{s-1}}{\binom{n-1}{s-1}}ip & \sim & \frac{1}{n}\sum\limits_{i=1}^{n}\sum\limits_{s=1}^{K}(n-s)\left(\frac{2(n-i)-(s-1)}{2(n-1)-(s-1)}\right)^{s-1} \cdot ip \\
& \leq & \frac{1}{n}\sum\limits_{i=1}^{n}\sum\limits_{s=1}^{K}(n-s)\frac{[2(n-1)-(s-1)]^{s+1}}{[2(n-1)-(s-1)]^{s-1}} \cdot \frac{1}{4s(s+1)} \cdot ip + \text{lower terms} \\
& = & \frac{1}{n}\sum\limits_{s=1}^{K}(n-s)\left(2(n-1)-(s-1)\right)^2 \cdot \frac{1}{4s(s+1)} \cdot p + \text{lower terms} \\
& \leq & \frac{1}{4n}\sum\limits_{s=1}^{K}(n-s)\left(2(n-1)\right)^2 \cdot \frac{1}{s(s+1)} \cdot p + \text{lower terms} \\
& = & \left((n-1)^2 \cdot \left(1 - \frac{1}{K+1}\right)\right) \cdot p + \text{lower terms}
\end{array}$$

where the first inequality is obtained by noting that $\sum\limits_{i=1}^{n} i(2(n-i))^{s-1}$ can be approximated by the integral $\int_1^n x(2(n-x))^{s-1}\,dx$, which results in $\frac{1}{4s(s+1)}[2(n-1)]^{s-1}$ and lower order terms. The above upperbound is maximized at $p = \frac{1}{n}$, so that we cannot do better than a constant factor of n.

□

Equivalently, the above result states that the expected rank of a hiree in the fixed K scenario is at least a constant factor of n.

6 Online Auction Incentive Compatibility

In certain settings, we need not be overly restrictive with incentive compatibility constraints $p_i = p_j$ for all $i, j \in \{1, \ldots, n\}$. Consider an online auction where potential bidders arrive in a random sequential order (think eBay), and we wish to give incentive to the bidder arriving at time slot i to bid right away, rather than delaying until a later time slot. It has been shown in [?] that for the classical setting, the optimal probability of selecting the best applicant is $\frac{1}{2\sqrt{e}} \approx 0.303265$. For the rank-based setting, we shall show that incentive compatibility in this online auction setting does not yield any asymptotic improvement over the traditional one.

Lemma 6.1. *Consider the rank-based secretary problem with incentive compatibility constraints $p_i \geq p_{i+1}$, for all i. The optimal expected rank for the rank-based version is also of order $\Theta(\log(n))$.*

Proof. Let $f(i,j) = \frac{n(i-j+1)-j}{i(i+1)}$, and observe that for a fixed i, $f(i,j)$ is decreasing in j. From the constraint $i \cdot p_i = \sum_{k=1}^{i} q_i^k$, and $f(i,j)$ monotonically decreasing in j, it follows that we want to shift as much weight into q_i^1 as possible.

Now consider a solution where all possible weights are shifted into q_i^1, leaving $q_i^2 = q_i^3 = \ldots = 0$, so that it has value $q_i^1 = i \cdot p_i$. The objective value for this particular solution is then $\sum_{i=1}^{n} \frac{ni-1}{i(i+1)} \cdot i \cdot p_i = \sum_{i=1}^{n} \frac{ni-1}{i+1} \cdot p_i$. Observe that $\frac{ni-1}{i+1}$ is increasing in i, so that we want to allocate more weight to p_j than to p_i whenever $j > i$. This, together with our incentive compatibile constraints $p_i \geq p_j$ whenever $j > i$ implies we must assign equal weights to all p_i's. Since $\sum_{i=1}^{n} p_i \leq 1$, we can only assign a maximum of $\frac{1}{n}$ to each p_i. The objective value for the **infeasible** solution $p_i = \frac{1}{n}$ and $q_i^1 = ip_i, q_i^2 = q_i^3 = \ldots = 0$ $\forall i$ is $n - \frac{n+1}{n} \sum_{i=1}^{n} \frac{1}{i+1}$. Since this is an upper bound to the expected utility problem, it follows that the optimal expected rank is of order $\Omega(\log(n))$.

Since $p_i \geq p_{i+1}$ for all i is a relaxation of the more restrictive $p_i = p_j$ for all i, j, and the optimal expected rank for the latter problem is in $\Theta(\log(n))$, our claim follows by having proved the optimal expected rank (for the version $p_i \geq p_{i+1}$) also belongs to $\Omega(\log(n))$.

□

7 A Concluding Note On Incentive Compatibility

In this paper, we showed the incentive compatibility property can be costly to the employer, with this greatly depending on her hiring objective. This result holds for both types of incentive compatibility considered in [?] and [?]. Furthermore, the optimal incentive compatible rank-based policy always make a hire, contrasting that of the optimal policy in the classical case. From a high-level perspective, we also showed Buchbinder, Jain, and Singh's linear programming based approach for modeling secretary problems can be obtained from known methods in the theory of Markov Decision Processes. Our insight allows for an alternative approach in cases where BJS's technique cannot be easily applied.

A LP Derivation For The Utility-Based Secretary Problem

The approach here is similar to that found in Buchbinder et al.'s paper [?]. The first step involves showing that all mechanisms must satisfy a certain set of linear constraints, and the corresponding linear program gives an objective value which is **at least** that of the mechanism's. The second step shows the converse, i.e. from a feasible solution to the linear program, construct a mechanism which selects an applicant with expected utility **at least** as high as that in the LP objective. These two steps then imply the problem of finding an optimal mechanism is equivalent to that of solving a particular linear program.

Lemma A.1. *Take any mechanism π for selecting applicants, while guaranteeing the incentive compatibility property. Let q_i^j denote the probability π selects the ith applicant given that she is jth best so far. Let p_i denote the probability π selects the ith applicant. Then the linear program below gives an upper bound to the expected utility of the applicant that π selects:*

$$\begin{array}{lll}
\textit{max} & \frac{1}{n}\sum_{i=1}^{n}\sum_{s=1}^{n}\sum_{j=1}^{s}(n-s)\frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}}q_i^j & \\
\textit{s.t.} & q_i^j \leq 1-\sum_{l<i} p_l & 1 \leq j \leq i \leq n \\
& p_i = \frac{1}{i}\sum_{k=1}^{i} q_i^k & 1 \leq i \leq n \\
& p_i = p & 1 \leq i \leq n \\
& q_i^j \geq 0 &
\end{array}$$

Proof. We shall first derive the objective function, and constraints afterward.

1. Let U be the random variable denoting the utility of hiring an applicant. Also define f_i^s to be the probability a mechanism π selects the ith applicant given she is sth best overall. Then:

$$\begin{aligned}
\mathbb{E}[U] &= \sum_{i=1}^{n}\sum_{s=1}^{n}\mathbb{E}[U \mid \pi \text{ selects } i,\ \pi \text{ did not select } 1,\ldots,i-1;\ \ i \text{ is } s\text{th best overall}] \\
&\qquad \cdot \Pr[\pi \text{ selects } i \mid \pi \text{ did not select } 1,\ldots,i-1;\ \ i \text{ is } s\text{th best overall}] \\
&\qquad \cdot \Pr[\pi \text{ did not select } 1,\ldots,i-1;\ \ i \text{ is } s\text{th best overall}] \\
&= \sum_{i=1}^{n}\sum_{s=1}^{n}(n-s)\cdot\Pr[\pi \text{ selects } i \mid i \text{ is } s\text{th best overall}]\cdot\Pr[i \text{ is } s\text{th best overall}] \\
&= \frac{1}{n}\sum_{i=1}^{n}\sum_{s=1}^{n}(n-s)\cdot f_i^s
\end{aligned}$$

Next, we will show $f_i^s = \sum_{j=1}^{s}\frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}}q_i^j$, which completes the argument. Observe that:

$$\begin{aligned}
f_i^s &= \Pr[\pi \text{ selects } i \mid i\text{th applicant is } s\text{th best overall}] \\
&= \sum_{j=1}^{s}\Pr[\pi \text{ selects } i \mid i\text{th applicant is } s\text{th best overall}, \\
&\qquad\qquad i\text{th applicant is } j\text{th best so far}] \\
&\qquad \cdot \Pr[i\text{th applicant is } j\text{th best so far} \mid i\text{th applicant is } s\text{th best overall}] \\
&= \sum_{j=1}^{s}\Pr[\pi \text{ selects } i \mid i\text{th applicant is } j\text{th best so far}] \\
&\qquad \cdot \Pr[i\text{th applicant is } j\text{th best so far} \mid i\text{th applicant is } s\text{th best overall}] \\
&= \sum_{j=1}^{s} q_i^j\cdot\frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}}
\end{aligned}$$

Here, the third equality follows because a mechanism can only discern at position i whether this applicant is jth best so far or not. The information that she is sth best overall is irrelevant.

To see the fourth equality, observe that out of $n-1$ positions, we must choose $s-1$ that are of lower rank than the ith position; there are $\binom{n-1}{s-1}$ ways to do this. For the numerator, with the ith element being jth best so far, among the first $i-1$ positions, choose $j-1$ to be of lower rank; there are $\binom{i-1}{j-1}$ ways. Also, among the other $n-i$ positions that come after the ith applicant, choose $s-j$ positions to be occupied by the rest of the applicants with smaller rank than the ith applicant. This gives $\binom{n-i}{s-j}$ ways, and completes the argument.

2.

$$\begin{aligned} q_i^j &= \Pr[\pi \text{ selects } i \mid i\text{th applicant is } j\text{th best so far}] \\ &\leq \Pr[\pi \text{ did not select } 1, 2, \ldots, i-1 \mid i\text{th applicant is } j\text{th best so far}] \\ &= 1 - \sum_{l<i} \Pr[\pi \text{ selects lth applicant} \mid i\text{th applicant is } j\text{th best so far}] \\ &= 1 - \sum_{l<i} \Pr[\pi \text{ selects lth applicant}] \\ &= 1 - \sum_{l<i} p_l \end{aligned}$$

3.

$$\begin{aligned} p_i &= \Pr[\pi \text{ selects } i] \\ &= \sum_{j=1}^{i} \Pr[\pi \text{ selects } i \mid i\text{th applicant is } j\text{th best so far}] \\ &\qquad\qquad \cdot \Pr[i\text{th applicant is } j\text{th best so far}] \\ &= \sum_{j=1}^{i} q_i^j \cdot \frac{1}{i} \end{aligned}$$

4. Since the probability of being selected is the same for all slots, no one has incentive to switch to a different interview slot. Thus this constraint restricts our search to incentive compatible strategies.

Since all incentive compatible hiring mechanisms need to satisfy these constraints, it follows that the optimal objective function is an upper bound to the expected utility of the hired applicant.

□

Lemma A.1. shows any mechanism π must satisfy feasibility for a particular linear program, and its performance is upper bounded by the objective function of that linear program. In the next lemma, we show how to construct a mechanism from a solution of the linear program. Together, these two lemmas show a one-to-one correspondence between mechanisms and LP feasible solutions, and thus completes the proof for one-to-one correspondence between LP's feasible solutions and hiring mechanisms.

Lemma A.2. *Let the triple (p, p_i, q_i^j) be a feasible solution to the linear program presented earlier. Consider the mechanism μ which, given it has not selected applicants $1, 2, \ldots, i-1$ and the ith applicant is jth best so far, picks the ith applicant with probability $\frac{q_i^j}{1-\sum_{l<i} p_l} = \frac{q_i^j}{1-(i-1)p}$. Then the expected utility of the hired applicant for which μ selected is that of the objective value:*

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{s=1}^{n} \sum_{j=1}^{s} (n-s) \frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}} q_i^j$$

Furthermore, $p_i^\mu = p_j^\mu = p$, so that when μ is employed as a hiring mechanism, every applicant is not envious of another's interview slot.

Proof. We first show the probability of μ selecting applicant i **given** that this ith applicant is jth best so far is q_i^j. Furthermore, the probability of μ selecting the ith applicant is $p_i = p$. We proceed by induction on i. Once proved, it is easily seen that μ selects an applicant with expected rank given by the objective value in the linear program, by the argument in the previous lemma.

- $i = 1$: This is trivially true, since no applicants appear before the 1st, and the 1st applicant must be the best so far, it follows that q_1^1 is the probability μ selects applicant 1 **given** applicant 1 is the best so far.
- $i \leq k$: Assume for all $1 \leq j \leq i \leq k$, q_i^j is the probability that μ selects the ith applicant **given** the ith applicant is jth best so far. Also assume p_i is the probability μ selects the ith applicant.

- $i = k+1$: Note that

$$\begin{aligned}
q_{k+1}^j &\overset{?}{=} \Pr[\mu \text{ selects } (k+1)\text{th applicant} \mid (k+1) \text{ is } j\text{th best so far}] \\
&= \Pr[\mu \text{ selects } (k+1)\text{th applicant} \mid 1,2,\dots,k \text{ not selected}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad (k+1)\text{th is } j\text{th best so far}] \\
&\qquad \cdot \Pr[\mu \text{ did not select } 1,2,\dots,k \mid (k+1)\text{th is } j\text{th best so far}] \\
&= \frac{q_{k+1}^j}{1-\sum_{l<k+1} p_l} \cdot \left(1 - \sum_{l<k+1} p_l\right) \\
&= q_{k+1}^j
\end{aligned}$$

where the third inequality comes from the inductive assumption. It is also clear that since $p_{k+1} = \sum_{j=1}^{k+1} q_{k+1}^j \cdot \frac{1}{n}$, the LHS denotes the probability of μ selecting the $(k+1)$th applicant. This completes our proof.

□

Thus, we have reduced the problem of finding the optimal incentive compatible hiring mechanism to that of solving a linear program. The next subsection shows how to simplify this linear program further to obtain the form as officially presented earlier.

A.1 Simplification of Linear Program

Proposition A.3. *For $1 \le j \le i \le n$:*

$$\sum_{s=j}^{n} (n-s) \frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}} = \frac{n^2(i-j+1) - nj}{i(i+1)}$$

Proof. First, observe that $\frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}}$ is the probability of the ith applicant being jth best so far **given** she is sth best overall. Using Bayes' rule, we obtain:

$$\begin{aligned}
\frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}} &= \Pr[i\text{th applicant is } j\text{th best so far} \mid i\text{th applicant is } s\text{th best overall}] \\
&= \Pr[i\text{th applicant is } s\text{th best overall} \mid i\text{th applicant is } j\text{th best so far}] \\
&\qquad \cdot \frac{\Pr[i\text{th applicant is } j\text{th best so far}]}{\Pr[i\text{th applicant is } s\text{th best overall}]} \\
&= \frac{\binom{s-1}{j-1}\binom{n-s}{i-j}}{\binom{n}{i}} \cdot \frac{n}{i}
\end{aligned}$$

Hence, it follows that:

$$\begin{aligned}
\sum_{s=j}^{n} (n-s) \frac{\binom{i-1}{j-1}\binom{n-i}{s-j}}{\binom{n-1}{s-1}} &= \sum_{s=j}^{n} (n-s) \frac{n}{i} \frac{\binom{s-1}{j-1}\binom{n-s}{i-j}}{\binom{n}{i}} \\
&= \frac{n^2}{i} \sum_{s=j}^{n} \frac{\binom{s-1}{j-1}\binom{n-s}{i-j}}{\binom{n}{i}} - \sum_{s=j}^{n} \frac{sn}{i} \frac{\binom{s-1}{j-1}\binom{n-s}{i-j}}{\binom{n}{i}} \\
&= \frac{n^2}{i} - \sum_{s=j}^{n} \frac{jn}{i} \frac{\binom{s}{j}\binom{n-s}{i-j}}{\binom{n}{i}} \\
&= \frac{n^2}{i} - \frac{jn}{i} \frac{\binom{n+1}{i+1}}{\binom{n}{i}} \\
&= \frac{n^2(i-j+1)-nj}{i(i+1)}
\end{aligned}$$

In the fourth equality above, we used the well-known combinatorial identity $\sum_{s=j}^{n} \binom{s}{j}\binom{n-s}{i-j} = \binom{n+1}{i+1}$.

□

We can now simplify the earlier linear program using the previous proposition and through exchanging a double sum. The result is what we have been after.

B LP Derivation For The Secretary Problem With Backward Solicitation

Lemma B.1. *Take any mechanism π for selecting applicants, while maintaining the incentive compatibility condition. Let $\alpha(s,i)$ be the probability for which the ith applicant will accept an offer at the sth round. Let $p_{s,i}$ denote the probability π selects the ith applicant in the sth round. Let p_s be the probability π selects someone in the sth round. Then the linear program below gives an upper bound to the probability that π selects the best applicant:*

$$
\begin{array}{lll}
\textit{max} & \frac{1}{n}\sum\limits_{s=1}^{n} s \cdot p_s & \\
\textit{s.t} & s \cdot p_{s,i} \leq \alpha(s,i) \cdot \left(1 - \sum\limits_{l<i} p_l - \sum\limits_{l \geq i}^{s-1} l \cdot p_{l,i}\right) & \forall\ 1 \leq i \leq s \leq n \\
 & p_s = \sum\limits_{i=1}^{s} p_{s,i} & \forall\ 1 \leq s \leq n \\
 & p_{s,i} \geq 0,\ \ p_s \ \textit{free} &
\end{array}
$$

Proof. Define the event $A_{s,i}$ to be such that π **offers** the ith applicant in the sth round. Define $B_{s,i}$ to be such that the ith applicant is available in the sth round. We denote $C_{s,i}$ as the intersection of events $A_{s,i}$ and $B_{s,i}$, and say this is the event where π **selects** the ith applicant in the sth round (i.e. π **offers** the ith applicant in the sth round *and* this ith applicant is available in the sth round). Let $D_{s,i}$ be the event the ith applicant is the relative best in round s (i.e. he is the best among the first s applicants).

Take $q_{s,i} = \Pr[A_{s,i}, B_{s,i} \mid D_{s,i}] = \Pr[C_{s,i} \mid D_{s,i}]$, i.e. the probability a policy π **selects** the ith applicant in sth round *given* that this ith applicant is the relative best out of first s.

Take $p_{s,i} = \Pr[A_{s,i}, B_{s,i}] = \Pr[C_{s,i}]$, i.e. the probability a policy **selects** the ith applicant in sth round. Observe that $s \cdot p_{s,i} = q_{s,i}$ by a simple conditioning (here we focus on policies that only select the best so far, due to our objective function).

Take $p_s = \sum\limits_{i=1}^{s} p_{s,i}$, i.e. the probability the policy π **selects** an applicant in the sth round.

Let us obtain an upperbound for $q_{s,i}$. We have:

$$
\begin{aligned}
q_{s,i} &= \Pr[\pi \text{ selects the } i\text{th applicant in } s\text{th round} \mid i\text{th applicant is best out of } s] \\
&= \Pr[\pi \text{ offers the } i\text{th applicant in } s\text{th round } \textit{and} \\
&\qquad\qquad i\text{th applicant is available at round } s \mid i\text{th applicant is best out of first } s] \\
&= \Pr[\pi \text{ offers the } i\text{th applicant in } s\text{th round} \mid i\text{th applicant is best out of first } s] \\
&\qquad\qquad \cdot \alpha(s,i) \\
&\leq \Pr[\pi \text{ did not select anyone in rounds } 1,2,\ldots,i-1 \textit{ and} \\
&\qquad \text{did not select } i\text{th applicant in rounds } i, i+1, \ldots, s-1 \\
&\qquad\qquad \mid i\text{th applicant is best out of first } s] \cdot \alpha(s,i) \\
&= \left(1 - \sum_{l=1}^{i-1} p_l - \sum_{l=i}^{s-1} l \cdot p_{l,i}\right) \cdot \alpha(s,i)
\end{aligned}
$$

Here, the third equality follows the fact that being available is independent of all other events. The inequality can be obtained by observing that in order to **offer** the ith applicant in round s, the policy must not have **selected** anyone in rounds $1, 2, \ldots, i-1$, and must not have **selected** the ith applicant in rounds $i, i+1, \ldots, s-1$ (due to the conditional that this ith applicant is best out of the first s).

The last equality requires a bit of justification. First, observe that the probability of a feasible policy selecting someone in the sth round and selecting someone in the tth round is 0 (we can only select one person in the entire process). As such we can decompose the right hand side into sums of probabilities of selecting someone at different rounds. Second, we must have $\Pr[\pi \text{ selects } i\text{th in round } j \mid i\text{th is best out of } j] = q_{j,i} = j \cdot p_{j,i}\ \forall j \geq i$.

Replacing $q_{s,i}$ with $s \cdot p_{s,i}$ and we obtain the desired inequality.

Consider the same policy π, we must have the following as the objective value:

$$\begin{aligned}
\Pr[\pi \text{ selects best overall}] &= \sum_{i=1}^{n} \Pr[\pi \text{ selects } i\text{th applicant} \mid i\text{th applicant is best overall}] \\
&\qquad \cdot \Pr[i\text{th applicant is best overall}] \\
&= \frac{1}{n} \sum_{i=1}^{n} \sum_{s=i}^{n} \Pr[\pi \text{ selects } i\text{th applicant in round } s \\
&\qquad\qquad \mid i\text{th applicant is best overall}] \\
&= \frac{1}{n} \sum_{s=1}^{n} \sum_{i=1}^{s} \Pr[\pi \text{ selects } i\text{th applicant in round } s \\
&\qquad\qquad \mid \text{he is best among first } s] \\
&= \frac{1}{n} \sum_{s=1}^{n} \sum_{i=1}^{s} q_{s,i} \\
&= \frac{1}{n} \sum_{s=1}^{n} s \cdot p_s
\end{aligned}$$

The third equality follows by observing that at round s, π cannot distinguish between whether this ith applicant is best overall or best among s. As such, these conditional probabilities are the same.

We have shown all selection policies must satisfy these constraints, and have probability of selecting the best applicant as given by the objective function. As such, the linear program's optimal value is an upperbound to the probability of selecting the best applicant in the secretary problem with backward solicitation. This also finishes the proof.

□

Lemma B.2. *From any feasible solution to the linear program presented earlier, we can construct a hiring mechanism which will allow us to select the best applicant with probability matching that of the objective function.*

Proof. Suppose we have p_s and $p_{s,i}$ satisfying constraints of the linear program. Let us construct the policy of hiring applicants from these values of p_s and $p_{s,i}$. Define this mechanism π so that it **offers** the ith applicant in round s **given** this ith applicant is best among the first s, it did not select anyone in rounds $1, 2, \ldots, i-1$ and did not select ith applicant in rounds $i, i+1, \ldots, s-1$ with probability $\frac{s \cdot p_{s,i}}{\alpha(s,i) \cdot \left(1 - \sum_{l<i} p_l - \sum_{l \geq i}^{s-1} l \cdot p_{l,i}\right)}$ (and 0 if $\alpha(s,i) = 0$). In other words, conditional on the process still going at round s and the ith applicant is the best at this stage, we should extend offer to this ith applicant with the above defined probability.

With the above constructed policy π, we claim π **selects** the ith applicant in round s with probability $p_{s,i}$, and (which follows directly from $p_{s,i}$) **selects** someone in round s with probability p_s. We shall show this by induction on s (the rounds).

- $s = 1$: then π offers the 1st applicant in round 1 with probability $\frac{1 \cdot p_{1,1}}{\alpha(1,1)}$. As such, π selects the 1st applicant in round 1 with probability $\frac{1 \cdot p_{1,1}}{\alpha(1,1)} \cdot \alpha(1,1) = p_{1,1}$.

- $s \leq \hat{s}$: assume the constructed policy π selects the ith applicant in round $\hat{s}$ with probability $p_{\hat{s},i}$ for all $1 \leq i \leq \hat{s}$, and selects someone in the sth round with probability p_s.

- $s = \hat{s}+1$: from the constructed policy π, observe that it **selects** the ith applicant in round $\hat{s}+1$ given that it did not select anyone in rounds $1, 2, \ldots, i-1$ and did not select ith applicant in rounds $i, i+1, \ldots, \hat{s}$, and the ith applicant is best out of $\hat{s}+1$, is $\frac{(\hat{s}+1) \cdot p_{\hat{s}+1,i}}{1 - \sum_{l<i} p_l - \sum_{l=i}^{\hat{s}} l \cdot p_{l,i}}$. Next, we can find the probability π selects the ith applicant in round $\hat{s}+1$ by conditioning on whether ith applicant is best so far at round $\hat{s}+1$, and whether the process can reach the stage $\hat{s}+1$. Multiplying out the respective probabilities gives us what we desire.

Now that we have shown p_s and $p_{s,i}$ are indeed probabilities of the constructed policy π selecting applicants, the probability π selecting the best overall applicant can be computed as in the proof of the previous **Lemma**. As such, we have shown (constructed) a policy from the LP solution which gives matching objective value.

$\square$

These two lemmas allow us to use the presented linear program for solving the secretary problem with backward solicitation.