Strategic Choices in the Secretary Problem

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Abstract

We consider two variants of the secretary problem where each applicant is allowed to specify a window of availability, whence the employer can only hire an applicant if he is available within that time frame. The difference between these two variants lies in when applicants make known of their windows of availability to the employer. In the first model, the applicant only makes known of his window of availability at the time of his interview, whereas in the second model all applicants will simultaneously announce their windows of availability to the employer before the job search begins. Furthermore, applicants' commitments are binding in the second model. The employer's objective is, as in the classical secretary problem, to maximize her probability of selecting the overall best applicant. For the sequential-move variant, we show strongly dominating strategies for applicants, and observe that the employer will select the best overall with the higher probability as that in the classical setting, and will converge from above as the number of applicants tend to infinity. For the simultaneous-move variant, we show how a pure-strategy equilibrium needs not exist for the case $n = 4$ applicants: its existence is dependent on the employer's optimal hiring policy. We prove existence of pure-strategy equilibria for all cases of $n \neq 4$, **independent** of the employer's optimal hiring policy. Furthermore, as $n \to \infty$, under these equilibria, the employer's probability of hiring the best applicant converges to $\frac{1}{2}$.

1 Introduction

In the classic secretary problem, n (commonly known) applicants present themselves one by one to an employer. If the employer was able to view them all at once, she would be able to rank them from best to worst. The order for which applicants present themselves to the employer is random, i.e. it is equally likely from among the $n!$ permutations. As an applicant presents himself, the employer must make her hiring decision on the spot: either to accept this applicant, or to reject him and moves on to the next applicant without the possibility of going back. The employer's objective is to design an algorithm that would maximize her probability of selecting the overall best applicant.

As shown in [**?**], the employer's optimal hiring policy is deceptively simple: to skip all applicants from slots 1 to r^*-1, and selects the applicant in slot $r^* \leq j \leq n$ if he happens to be the best among those observed so far. Here, r^* is the smallest integer between 1 and n which satisfies $\frac{r^*}{n} \geq \frac{r^*}{n} \sum\limits_{k=r^*+1}^{n} \frac{1}{k-1}$. This threshold policy is typical in optimal stopping problems, where the decision maker spends some time *learning* about the population, and starts working on the choosing process after a certain time point. A moment of reflection, however, seems to indicate that the approach may not work with all subjects. When we are dealing with items, this policy is perfectly fine, as items do not think strategically. When we are dealing with living, rational human beings, this policy begs for much improvement. The optimal threshold policy in the classical setting would interview and never select early applicants, giving them incentive to not participate in the process at all. This leads to a potential problem for the employer: she may not be able to observe and learn from the population to make a decision on her hiring choice. One approach that tries to mitigate this concern was proposed by Buchbinder et. al [**?**], and was extended and generalized to the rank-based case in [**?**]. There, the employer constrained herself to only hiring policies for which applicants have incentives to remain in their interview slots. Here, we shall allow applicants more freedom in the hiring process, where they can make decisions which will affect everyone involved, from other applicants to the employer herself.

In reality, one can improve his odds of getting hired by learning a new skill set, embarking on an internship which will provide him with valuable and relevant experience, or by simply bribing the hiring manager. Moving away from these slot-independent ideas, we choose to focus on a set of strategies which is dependent on one's interview slot position. The resulting mathematical problem becomes quite interesting, and gives us insights into how applicants may, and should, behave.

2 A Sequential Secretary Game: It Pays To Be Nice!

During each of the n days, the employer would invite an applicant to come in for an interview. After his interview has been conducted, applicant i tells his potential employer of a time window (window of availability) t_i where she must make her decision. For example, if the applicant is interviewed on the 4th day and his time window $t_4 = 0$, then the employer must make her decision for this applicant right away. Whereas if his time window is given as $t_4 = 3$, then the employer can make her decision on or before the 7th day. All applicants know their respective interview slots, and it is common knowledge that the employer wants to maximize her probability of selecting the best candidate.

Theorem 2.1. *When being asked for an interview time window, those in slots $1 \leq i \leq r^*$ will choose to allow the employer until time r^* to make her decision, i.e. $t_i = r^* - i$ for $1 \leq i \leq r^*$. Those in slots $r^* + 1 \leq i \leq n$ will force the employer to make her decision right away, i.e. $t_i = 0$ for these $r^* + 1 \leq i \leq n$. Here, r^* is the smallest integer between 1 and n such that $1 \geq \sum\limits_{k=r^*+1}^{n} \frac{1}{k-1}$.*

Proof. The proof is by backward induction. Since the last applicant has no choice but to use $t_n = 0$, the statement trivially holds in this case. If the employer ever gets to the $(n-1)$st applicant, then this applicant has two choices: either $t_{n-1} = 0$ or $t_{n-1} = 1$. We need only focus on the scenario that he is the best so far out of the first $n - 1$ applicants, as he will not get selected otherwise. Letting $t_{n-1} = 0$ will get him selected with probability 1, whereas letting $t_{n-1} = 1$ will get him selected with probability strictly smaller than 1 as the last applicant may be a better candidate than he is. As such, he will play $t_{n-1} = 0$.

The same argument holds for applicants in slot positions $r^* \leq i \leq n$. Again focusing on the best-so-far scenario, when the ith applicant chooses $t_i = 0$, he'd get selected with probability 1. For other strategies, his probability of getting chosen is strictly less than 1. As such, these applicants will also play $t_i = 0$.

Now consider the $(r^* - 1)$th applicant (again assuming he is the best so far). When he comes in for the interview, he knows later applicants will all play $t_j = 0$ for $j \geq r^*$. As such, if he lets $t_{r^*-1} = 0$, his chance of getting hired is 0. If he allows the time window to be $t_{r^*-1} = 1$, then the employer will make her hire at time r^*, and his chance of getting hired is $\frac{r^*-1}{r^*}$. If he lets $t_{r^*-1} = t > 1$, his probability of getting selected is at most $\frac{r^*-1}{r^*-1+t}$, which is strictly smaller than $\frac{r^*-1}{r^*}$. Therefore, he will play $t_{r^*-1} = 1$.

A similar argument can be applied for all applicants $1 \leq i \leq r^* - 2$. Allowing a time window deadline before r^* will not get them selected, whereas allowing a time window deadline after r^* will get them selected with lower probability than one at exactly r^*. As such, all applicants $1 \leq i \leq r^*$ will have $t_i = r^* - i$.

□

Corollary 2.2. *Under the sequential-move model, the employer will hire the best applicant with probability $\frac{r^*}{n}$, where r^* is the smallest integer between 1 and n such that $1 \geq \sum\limits_{k=r^*+1}^{n} \frac{1}{k-1}$. This is better than, and will converge from above to, her probability of selecting the best applicant under the classical, non-game-theoretic setting.*

Proof. Because applicants $1 \leq i \leq r^*$ all allow the employer until time r^* to make her hiring decision, she will choose to do so at time r^*. This gives her a probability of $\frac{r^*}{n}$ for picking the best overall applicant. Furthermore, from the definition of r^*, we obtain the following:

$$\begin{aligned}
& \sum_{k=r^*+1}^{n} \frac{1}{k-1} &\leq\ & 1 \\
\iff\ & \sum_{k=r^*+1}^{n} \frac{1}{k-1} + \frac{1}{r^*-1} &\leq\ & 1 + \frac{1}{r^*-1} \\
\iff\ & \sum_{k=r^*}^{n} \frac{1}{k-1} &\leq\ & \frac{r^*}{r^*-1} \\
\iff\ & \frac{r^*-1}{n} \sum_{k=r^*}^{n} \frac{1}{k-1} &\leq\ & \frac{r^*}{n}
\end{aligned}$$

Observe that the left hand side denotes the probability of selecting the best applicant under the classical setting, whereas the right hand side is one for our discussed game-theoretic setting. The statement of convergence is a simple derivation using integral calculus.

□

We observe that our assumption of having applicants submitting time windows right after their interviews can be weakened, and the result above will still hold. Suppose applicants can change their mind, and submit different windows of availability than ones given right after their interviews, or not submitting anything at all in the first place, they would still want to play the dominant strategies outlined in the theorem. As such, whenever the hiring process occurs sequentially, applicants would behave as we prescribed.

When the employer asks for all applicants' windows of availability at the beginning of a hiring process, before the first interview begins, the ensuing actions greatly depend on whether time window commitments are binding or not. If they are not binding, then we are back to our already discussed scenario. If they are binding, then we have a simultaneous game setting, which will be analyzed in the next section.

Finally, we recall that the optimal policy in the classical secretary setting is unfair to early applicants: they are interviewed but will not get hired. This fact became Buchbinder et. al.'s [?] main motivation for considering incentive compatible policies, where all interviewed subjects are guaranteed the same probability of getting hired. By allowing applicants more freedom of choice, our model solves this problem of never hiring an interviewed subject, and improves on the probability of selecting the best applicant from both [?] and the classical setting. It pays to be nice!

3 A Simultaneous Secretary Game: Binding Contracts And Equilibria

Continuing with our discussion from the last section, we next focus on the setting where all applicants are required to make known of their time windows before the first interview begins. These commitments are binding once submited, so that they cannot be changed later on, and each applicant must adhere to what he proposes. From these time windows, the employer can now use a corresponding optimal policy to maximize her probability of selecting the best applicant. We provide further details for this simultaneous game below.

3.1 Model And Payoff Matrix

When an applicant is assigned an interview slot by the employer, he must also make known his window of availability t_i. So that an applicant in slot i will be available for hire during times $i+0, i+1, \ldots, i+t_i$. Given the strategy $\vec{t} = (t_1, t_2, \ldots, t_n)$ played by the applicants, the employer can devise an optimal hiring mechanism $\pi(\vec{t})$, which when used, will give the probability $P_i^{\pi(\vec{t})}$ for selecting the applicant at position i. We say the strategy $\vec{t}$ is an equilibrium when, for every applicant i, we have $P_i^{\pi(\vec{t})} \geq P_i^{\pi(t,\vec{t}_{-i})} \quad \forall\ t \neq t_i$.

Computing the optimal hiring mechanism given $\vec{t}$ can be done through a linear program presented earlier:

$$
\begin{array}{llll}
\max & \frac{1}{n}\sum_{s=1}^{n} s\cdot p_s & & \\
\text{s.t.} & s\cdot p_{s,i} & \leq & \alpha(s,i)\left(1-\sum_{l<i} p_l - \sum_{l\geq i}^{s-1} l\cdot p_{l,i}\right) \\
& p_s & = & \sum_{i=1}^{s} p_{s,i} \\
& p_{s,i}\geq 0,\ p_s \text{ free} & &
\end{array}
$$

In the above, $\alpha(s,i)$ denotes the probability the ith applicant will accept an offer at time s (so that $\alpha(s,i)$ are data), and $p_{s,i}$ is the probability the employer chooses the ith applicant at time s. As such, given $\vec{t}$, we have the following: if $i+t_i\leq s-1$, then $\alpha(s,i)=0$; otherwise, $\alpha(s,i)=1$. An alternative way is by setting $\alpha(s,i)=1$ for all $1\leq i\leq s\leq n$. Given $\vec{t}$, if $i+t_i\leq s-1$, then set $p_{s,i}=0$.

Given $\vec{t}$, the employer can compute her optimal policy using the above linear program. From the perspective of the ith applicant, he will get chosen with probability $P_i^{\pi(\vec{t})}=\sum_{s=i}^{n} p_{s,i}^{\pi(\vec{t})}$. The problem can now be analyzed as a standard simultaneous game.

3.2 Pure-Strategy Equilibrium For The Case of 3 Applicants

Let us illustrate this for the case $n=3$, i.e. there are 3 applicants for the position. Consider the following feasible strategies $\vec{t}=(t_1,t_2,t_3)$.

1. $\vec{t}=(0,0,0)$: This is the classical setting. Solving yields the employer's probability of selecting the best applicant is $\frac{1}{2}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3
1	0	$\cdot$	$\cdot$
2	0	$\frac{1}{2}$	$\cdot$
3	0	0	$\frac{1}{6}$
$P_i^{\pi(\vec{t})}$	0	$\frac{1}{2}$	$\frac{1}{6}$

2. $\vec{t}=(1,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{2}{3}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3
1	0	$\cdot$	$\cdot$
2	$\frac{1}{2}$	$\frac{1}{2}$	$\cdot$
3	0	0	0
$P_i^{\pi(\vec{t})}$	$\frac{1}{2}$	$\frac{1}{2}$	0

3. $\vec{t}=(2,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{5}{6}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3
1	0	$\cdot$	$\cdot$
2	0	$\frac{1}{2}$	$\cdot$
3	$\frac{1}{3}$	0	$\frac{1}{6}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

4. $\vec{t}=(0,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{2}{3}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3
1	0	$\cdot$	$\cdot$
2	0	0	$\cdot$
3	0	$\frac{1}{3}$	$\frac{1}{3}$
$P_i^{\pi(\vec{t})}$	0	$\frac{1}{3}$	$\frac{1}{3}$

5. $\vec{t} = (1, 1, 0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{5}{6}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3
1	0	$\cdot$	$\cdot$
2	$\frac{1}{2}$	0	$\cdot$
3	0	$\frac{1}{3}$	$\frac{1}{6}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

6. $\vec{t} = (2, 1, 0)$: Solving yields the employer's probability of selecting the best applicant is 1, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3
1	0	$\cdot$	$\cdot$
2	0	0	$\cdot$
3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

From the above, it follows that $(1, 0, 0)$ is the only pure-strategy equilibrium, and the employer's probability for selecting the best applicant has improved to $\frac{2}{3}$.

3.3 Non-Existence of Pure-Strategy Equilibria

Having seen how there exists a pure-strategy equilibrium for the case $n = 3$ applicants, we naturally seek to know whether there exists a pure-strategy equilibrium for every secretary game with $n \geq 4$ applicants. The answer turns out to be negative, and is stated in the next theorem.

Theorem 3.1. *When there are $n = 4$ applicants, dependent on the employer's optimal hiring policy, the secretary hiring game may not have any pure-strategy equilibrium.*

Proof. The proof is by brute force inspection. The reader is invited to check all 24 cases of playable strategies for applicants. These payoff matrices can be found in the **Appendix**.

□

Existence of a pure-strategy equilibrium in the case where $n = 4$ is dependent on how the employer optimally selects applicants. If her optimal policy is as presented in the **Appendix**, then no pure-strategy equilibria exist. There, observe that when applicants play the strategies $(1, 0, 1, 0)$, the employer waits until time 4 to make a decision. But her probability of selecting the best applicant is also maximized had she chosen to make a decision at time 2, where she could choose between applicants 1 and 2. Had she followed this optimal hiring policy, then it can easily be checked that $(1, 0, 0, 0)$ is a pure-strategy equilibrium.

Having made this observation, a natural question is whether there exists a pure-strategy equilibrium for cases of $n \geq 5$, **independent** of how the employer optimally makes her hiring decision. Our next objective is to present such pure-strategy equilibria. It turns out that the structure of a pure-strategy equilibrium in the case of n odd is different from that in the case of n even. Since n odd is easier to state and prove, we shall start there.

3.4 Existence of Pure-Strategy Equilibria For $n \geq 3$, Odd

For this section, it is useful to focus on the strategy $\vec{t}$, where

$$t_i = \begin{cases} \lceil \frac{n}{2} \rceil - i & \text{for } 1 \leq i \leq \lceil \frac{n}{2} \rceil \\ n - i & \text{for } \lceil \frac{n}{2} \rceil + 1 \leq i \leq n \end{cases}$$

In other words, those in slots up to $\lceil \frac{n}{2} \rceil$ will allow the employer up to time $\lceil \frac{n}{2} \rceil$ to make her decision. Whereas those applicants in later slots will allow the employer until the end to make her decision. We shall show when n is odd, $\vec{t}$ constitutes a pure-strategy equilibrium.

Proposition 3.2. *Assuming n is odd. When all applicants adhere to playing $\vec{t}$, the employer selects the best applicant with probability $\frac{\lceil n/2 \rceil}{n} > \frac{1}{2}$. Each applicant in slots up to $\lceil \frac{n}{2} \rceil$ will get selected with probability $\frac{1}{\lceil n/2 \rceil}$. The rest of the applicants will not get selected.*

Proof. The employer will only make her hiring decision at one of two epochs: time $\lceil \frac{n}{2} \rceil$, or time n. In the former, her probability of selecting the best applicant is $\frac{\lceil n/2 \rceil}{n} > \frac{1}{2}$ (because n is assumed to be odd). In the latter, her probability of selecting the best applicant is $1 - \frac{\lceil n/2 \rceil}{n} < \frac{1}{2}$. As such, she will make her decision at time $\lceil \frac{n}{2} \rceil$, and only selects from among these $\lceil \frac{n}{2} \rceil$ applicants. It is clear that each of these applicants gets selected with probability $\frac{1}{\lceil n/2 \rceil}$.

□

The next lemma shows that if all other applicants adhere to playing the strategy $\vec{t}_{-i}$, then applicant i, where $1 \leq i \leq \lceil \frac{n}{2} \rceil$, does not want his window of availability to end before time $\lceil \frac{n}{2} \rceil$. Note that if $\lceil \frac{n}{2} \rceil < i \leq n$, then however he changes his strategy will not matter, as the employer will always choose at time point $\lceil \frac{n}{2} \rceil$, and therefore this applicant who comes after time $\lceil \frac{n}{2} \rceil$ will not get selected.

Lemma 3.3. *Consider an arbitrary applicant i, where $1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$, and suppose he changes his window of availability to end before time $\lceil \frac{n}{2} \rceil$. In other words, his new t'_i is such that $i + t'_i \leq \lceil \frac{n}{2} \rceil - 1$. If all other applicants adhere to the strategy $\vec{t}_{-i}$, then applicant i will get selected with probability* 0.

Proof. Let us denote the last time where applicant i is available as $\tilde{t}$; that is, $i + t'_i = \tilde{t}$. Due to the employer's objective, she must use a policy of hiring the best so far starting at one of three possible time epochs: $\tilde{t}$, $\lceil \frac{n}{2} \rceil$, or n. Let Π_α (Π_β) denote her policy of hiring the best so far starting at $\tilde{t}$ ($\lceil \frac{n}{2} \rceil$). Consider the following three possible scenarios.

- ith applicant is best out of $\tilde{t}$, but not best out of $\lceil \frac{n}{2} \rceil$.
  - This event happens with probabilty $\frac{1}{\tilde{t}} - \frac{1}{\lceil n/2 \rceil}$.
  - $\Pr[\Pi_\alpha \text{ selects best applicant}] = 0$.
  - $\Pr[\Pi_\beta \text{ selects best applicant}] = \frac{\lceil n/2 \rceil}{n}$.
- ith applicant is best out of $\tilde{t}$, and is best out of $\lceil \frac{n}{2} \rceil$.
  - This event happens with probability $\frac{1}{\lceil n/2 \rceil}$.
  - $\Pr[\Pi_\alpha \text{ selects best applicant}] = \frac{\lceil n/2 \rceil}{n}$.
  - $\Pr[\Pi_\beta \text{ selects best applicant}] = 1 - \frac{\lceil n/2 \rceil}{n}$.
- ith applicant is not the best out of $\tilde{t}$.
  - This event happens with probability $1 - \frac{1}{\tilde{t}}$.
  - $\Pr[\Pi_\alpha \text{ selects best applicant}] = \frac{\lceil n/2 \rceil}{n}$.
  - $\Pr[\Pi_\beta \text{ selects best applicant}] = \frac{\lceil n/2 \rceil}{n}$.

From these, it follows that:

$$\begin{aligned}\Pr[\Pi_\alpha \text{ selects best applicant}] &= \frac{\lceil n/2\rceil}{n}\cdot\frac{1}{\lceil n/2\rceil}+\frac{\lceil n/2\rceil}{n}\cdot\left(1-\frac{1}{\tilde{t}}\right)\\ &= \frac{1}{n}+\frac{\lceil n/2\rceil}{n}-\frac{\lceil n/2\rceil}{\tilde{t}n}\end{aligned}$$

and

$$\begin{aligned}\Pr[\Pi_\beta \text{ selects best applicant}] &= \frac{\lceil n/2\rceil}{n}\cdot\frac{1}{\tilde{t}}-\frac{1}{n}+\frac{1}{\lceil n/2\rceil}-\frac{1}{n}+\frac{\lceil n/2\rceil}{n}-\frac{\lceil n/2\rceil}{\tilde{t}n}\\ &= \frac{1}{\lceil n/2\rceil}+\frac{\lceil n/2\rceil}{n}-\frac{2}{n}\end{aligned}$$

Observe that $\Pr[\Pi_\alpha \text{ selects best applicant}] < \Pr[\Pi_\beta \text{ selects best applicant}]$ if and only if $\frac{3}{n} < \frac{1}{\lceil n/2\rceil}+\frac{\lceil n/2\rceil}{\tilde{t}n}$, which holds for all odd $n \geq 3$, and $\tilde{t} \leq \lceil \frac{n}{2}\rceil - 1$.

As such, the employer will never use policy Π_α, and hence applicant i will never get selected. This proves the lemma.

□

Although not needed for the proof above, we note that the policy of starting the selection at time n is also dominated by Π_β. It is not needed because that policy also does not select the ith applicant, so that he does not have incentive to change his window of availability.

The next lemma shows that it also does not pay for an applicant to hold off until a time beyond $\lceil \frac{n}{2}\rceil$.

Lemma 3.4. *Consider an arbitrary applicant i, where $1 \leq i \leq \lceil \frac{n}{2}\rceil$, and suppose he changes his window of availability to end after time $\lceil \frac{n}{2}\rceil$. In other words, his new t'_i is such that $i + t'_i \geq \lceil \frac{n}{2}\rceil + 1$. If all other applicants adhere to the strategy $\vec{t}_{-i}$, then applicant i will get selected with probability strictly less than $\frac{1}{\lceil n/2\rceil}$.*

Proof. Let us denote the last time where applicant i is available as $\tilde{t}$; that is, $i + t'_i = \tilde{t}$. Due to the employer's objective, she must use a policy of hiring the best so far starting at one of three possible time epochs: $\lceil \frac{n}{2}\rceil$, $\tilde{t}$, or n. Let Π_α (Π_β) denote her policy of hiring the best so far starting at $\tilde{t}$ ($\lceil \frac{n}{2}\rceil$). Consider the following three possible scenarios: (1) the ith applicant is best out of $\lceil \frac{n}{2}\rceil$, but not best out of $\tilde{t}$; (2) the ith applicant is best out of $\lceil \frac{n}{2}\rceil$, and also is best out of $\tilde{t}$; and (3) the ith applicant is not the best out of $\lceil \frac{n}{2}\rceil$. Observe that both Π_α and Π_β select the best applicant with probability 1 in case (1), and that they both select the best applicant with probability $\frac{\tilde{t}}{n}$ in case (2). In case (3), the probability is $1 - \frac{\lceil n/2\rceil}{n}$ for Π_α, which is strictly less than the probability of $\frac{\lceil n/2\rceil}{n}$ for Π_β. As such, the employer will choose to use Π_β.

Using a similar argument, it can be shown that the policy of waiting until time n to make a decision is strongly dominated by Π_β as well. As such, policy Π_β will be used.

Given that the employer uses Π_β to optimally select the best candidate, the ith applicant will only get hired in scenario (2), which happens with probability $\frac{1}{\tilde{t}}$. Observe that this probability is strictly less than $\frac{1}{\lceil n/2\rceil}$, which is the probability the ith applicant would have been selected had he chosen to play $\tilde{t} = \lceil n/2\rceil$. Therefore, he will not delay to beyond time $\lceil n/2\rceil$.

□

The previous two lemmas allow us to conclude with the following theorem.

Theorem 3.5. *With $n \geq 3$ and n being odd, then $\vec{t}$ is a pure-strategy equilibrium.*

Proof. For $1 \leq i \leq \lceil \frac{n}{2}\rceil$, applicant i does not want to change his strategy since his probability of getting selected will be lowered. For $\lceil \frac{n}{2}\rceil < i \leq n$, applicant i changing his strategy does not matter as his probability of getting selected still remains at 0. As such, $\vec{t}$ is a pure-strategy equilibrium.

□

As was observed for the case $n = 4$, this constructed strategy is not always a pure-strategy equilibrium for even n. Whether it is an equilibrium or not depends on how the employer optimally makes her hiring decision. Given $\vec{t}$, she has equal probability of selecting the best applicant (at $\frac{1}{2}$) by either choosing at $\lceil \frac{n}{2}\rceil$, or at n. If she makes her hiring decision at $\lceil \frac{n}{2} \rceil$, then $\vec{t}$ is a pure-strategy equilibrium by arguments in the previous theorem. If, on the other hand, she makes her hiring decision at n, then $\vec{t}$ is not an equilibrium due to early applicants having a better chance of getting selected by changing their windows of availability to end at n (from 0 to $\frac{1}{n}$).

Thus, we now place our focus on finding a pure Nash equilibrium for $n \geq 6$, n being even, independent of the employer's optimal hiring policy.

3.5 Existence of Pure-Strategy Equilibria For $n \geq 6$, Even

In this subsection, we will show there exists a pure-strategy equilibrium for $n \geq 6$, and n being even, independent of the employer's optimal hiring policy. The proof is constructive, and as such, it is useful to define $\vec{t}$ as the strategy where

$$t_i = \begin{cases} \frac{n}{2} - i & \text{for } 1 \leq i \leq \frac{n}{2} \\ n - 1 - i & \text{for } \frac{n}{2} + 1 \leq i \leq n - 1 \\ 0 & \text{for } i = n \end{cases}$$

In other words, under the collective strategy $\vec{t}$, the first half of applicants will all allow the employer until time $\frac{n}{2}$ to make her hiring decision. The second half of applicants (except the last one) will allow the employer until time $n-1$ to make her hiring decision. And the applicant in the nth slot has no other choice but to remain in that position.

Proposition 3.6. *When all applicants adhere to playing $\vec{t}$, the employer selects the best applicant with probability $\frac{1}{2}$. Each applicant in slots up to $\frac{n}{2}$ will get selected with probability $\frac{2}{n}$. The rest of the applicants will not get selected.*

Proof. When n is even, if the employer selects the best one at time $\frac{n}{2}$, then her probability of selecting the best applicant is $\frac{n/2}{n} = \frac{1}{2}$. If she waits until a later time, then her probability of selecting the best applicant is strictly less than $\frac{1}{2}$, as there are $\frac{n}{2}$ applicants left, and at time $n-1$ she does not know the total order of all applicants (the last applicant has yet to appear). As such, she will make her hiring decision at time $\frac{n}{2}$. Under this optimal hiring policy, observe that each applicant $1 \leq i \leq \frac{n}{2}$ gets selected with probability $\frac{1}{n/2}$. The others are selected with probability 0.

□

The next lemma shows that if all other applicants adhere to playing the strategy $\vec{t}_{-i}$, then applicant i, where $1 \leq i \leq \frac{n}{2}$, does not want his window of availability to end before time $\frac{n}{2}$. Note that if $\frac{n}{2} < i \leq n$, then however he changes his strategy will not matter, as he will not get selected (holds for n even, and $n \geq 6$; in the case $n = 4$, the third applicant delaying to time 4 will allow the employer to be able to select the best applicant with probability $\frac{1}{2}$ by hiring at time 4).

Lemma 3.7. *Consider an arbitrary applicant i, where $1 \leq i \leq \frac{n}{2} - 1$, and suppose he changes his window of availability to end before time $\frac{n}{2}$. In other words, his new t'_i is such that $i + t'_i \leq \frac{n}{2} - 1$. If all other applicants adhere to the strategy $\vec{t}_{-i}$, then applicant i will get selected with probability* 0.

Proof. Denote $\tilde{t} = i + t'_i$, so that $\tilde{t}$ is the last time applicant i is available for hire. Although $\lceil \frac{n}{2} \rceil$ is equivalent to $\frac{n}{2}$ when n is even, we will use the former notation throughout this proof to illustrate a point: that this proposed strategy is not an equilibrium when n is odd. Now, observe that the employer will only make her hiring decisions starting at one of the time epochs $\tilde{t}$, $\lceil \frac{n}{2} \rceil$, $n-1$, or n. Let Π_α be the policy where the employer hires the best so far starting at time $\tilde{t}$. Also let Π_β be the policy where the employer hires the best so far starting at time $\lceil \frac{n}{2} \rceil$. We first show that the employer will choose applicants with the policy Π_β. Consider the following three scenarios:

- ith applicant is best at time $\tilde{t}$, but is not best at time $\lceil \frac{n}{2} \rceil$.
  - This event occurs with probability $\frac{1}{\tilde{t}} - \frac{1}{\lceil n/2 \rceil}$.
  - $\Pr[\Pi_\alpha \text{ selects best} \mid \text{event}] = 0$. This is because the policy Π_α would have already picked the ith applicant at time $\tilde{t}$, which is not the best overall.

- $\Pr[\Pi_\beta \text{ selects best} \mid \text{event}] = \frac{\lceil n/2 \rceil}{n}$. This is because the policy Π_β would have been able to pick the best so far at time $\lceil \frac{n}{2} \rceil$.

- ith applicant is best at time $\tilde{t}$, and is also best at time $\lceil \frac{n}{2} \rceil$.
  - This event occurs with probability $\frac{1}{\lceil n/2 \rceil}$.
  - $\Pr[\Pi_\alpha \text{ selects best} \mid \text{event}] = \frac{\lceil n/2 \rceil}{n}$. This is because the policy Π_α would have picked the ith applicant at time $\tilde{t}$.
  - We need to decompose this event further into three smaller sub-events.
    1. ith applicant is also best so far at time $n-1$, and is also best so far at time n.
       * This event occurs with probability $\frac{1}{n}$.
       * $\Pr[\Pi_\beta \text{ selects best} \mid \text{smaller event}] = 0$. This is because the policy Π_β started hiring at time $\lceil \frac{n}{2} \rceil$, for which the ith applicant is already unavailable.
    2. ith applicant is also best so far at time $n-1$, but not best so far out of n.
       * This event occurs with probability $\frac{1}{n-1} - \frac{1}{n}$.
       * $\Pr[\Pi_\beta \text{ selects best} \mid \text{smaller event}] = 1$. This is because no applicants available for hire up until time $n-1$ is the best so far. As such, only at time n can Π_β hires the best so far, which turns out to be the best overall as well.
    3. ith applicant is not the best so far at time $n-1$.
       * This event occurs with probability $\frac{1}{\lceil n/2 \rceil} - \frac{1}{n-1}$.
       * $\Pr[\Pi_\beta \text{ selects best} \mid \text{smaller event}] = \frac{n-1}{n}$. This is because the best so far is not available at time $\lceil \frac{n}{2} \rceil$, but is available at time $n-1$.

- ith applicant is not best at time $\tilde{t}$.
  - This event occurs with probability $1 - \frac{1}{\tilde{t}}$.
  - $\Pr[\Pi_\alpha \text{ selects best} \mid \text{event}] = \frac{\lceil n/2 \rceil}{n}$. This is because the best so far will surely be available at time $\lceil \frac{n}{2} \rceil$.
  - $\Pr[\Pi_\beta \text{ selects best} \mid \text{event}] = \frac{\lceil n/2 \rceil}{n}$. The reasoning is the same as above.

From these, we observe that:

$$\begin{aligned}
\Pr[\Pi_\alpha \text{ selects best}] < \Pr[\Pi_\beta \text{ selects best}] \quad &\iff \quad \frac{1}{n} < \frac{\lceil n/2 \rceil}{n\tilde{t}} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n} + \frac{n-1}{n\lceil n/2 \rceil} - \frac{1}{n} \\
&\iff \quad \frac{4}{n} - \frac{1}{n-1} - \frac{n-1}{n\lceil n/2 \rceil} < \frac{\lceil n/2 \rceil}{n\tilde{t}}
\end{aligned}$$

For $i \leq \tilde{t} \leq \lceil \frac{n}{2} \rceil - 1$, the right hand side is smallest at $\tilde{t} = \lceil \frac{n}{2} \rceil - 1$. We shall check for the strict inequality for this value of $\tilde{t}$. This strict inequality will be shown to **hold** for the case where n is even, and **fails** in the case where n is odd. As such, this strategy is only a candidate for equilibrium when there are an even number of applicants.

1. $n = 2k$, for $k \geq 1$.

$$\begin{aligned}
\frac{4}{n} - \frac{1}{n-1} - \frac{n-1}{n\lceil n/2 \rceil} < \frac{\lceil n/2 \rceil}{n\tilde{t}} \quad &\iff \quad \frac{4}{2k} - \frac{1}{2k-1} - \frac{2k-1}{2k^2} < \frac{k}{2k\cdot(k-1)} \\
&\iff \quad \frac{2}{k} - \frac{1}{2k-1} - \frac{1}{k} + \frac{1}{2k^2} < \frac{1}{2(k-1)} \\
&\iff \quad \frac{2(k-1)}{k} - \frac{2k-2}{2k-1} + \frac{k-1}{k^2} < 1 \\
&\iff \quad \frac{1}{2k-1} < \frac{1}{k}\left(1 + \frac{1}{k}\right) \\
&\iff \quad 1 < \left(2 - \frac{1}{k}\right)\left(1 + \frac{1}{k}\right)
\end{aligned}$$

But this strict inequality is obviously true for all $k \geq 1$. Hence the employer will use the policy Π_β, and as such the ith applicant gets selected with probability 0. Since his probability of getting selected had he stayed with the strategy t_i is $\frac{1}{n/2}$, he does not have incentive to play t_i'.

2. $n = 2k+1$, for $k \geq 2$.

$$\begin{aligned}
\tfrac{4}{n} - \tfrac{1}{n-1} - \tfrac{n-1}{n\lceil n/2\rceil} < \tfrac{\lceil n/2\rceil}{n\tilde{t}} \quad &\iff \quad \tfrac{4}{2k+1} - \tfrac{1}{2k} - \tfrac{2k}{(2k+1)(k+1)} < \tfrac{k+1}{(2k+1)\cdot k} \\
&\iff \quad 4 - \tfrac{2k+1}{2k} - \tfrac{2k}{k+1} < \tfrac{k+1}{k} \\
&\iff \quad 2 < \tfrac{3/2}{k} + \tfrac{2k}{k+1} \\
&\iff \quad 2 < \tfrac{(3/2)(k+1)+2k^2}{k(k+1)}
\end{aligned}$$

which fails for $k \geq 4$.

And since Pr[Π_α selects best] $>$ Pr[Π_β selects best] when an arbitrary applicant i shortens his window of availability to $\tilde{t} = \lceil \frac{n}{2} \rceil - 1$, the employer will use policy Π_α in this instance. But that implies the applicant i with $\tilde{t}$ will get selected in the first two scenarios for a total probability of $\frac{1}{\tilde{t}} - \frac{1}{\lceil n/2\rceil} + \frac{1}{\lceil n/2\rceil} = \frac{1}{\tilde{t}} > \frac{1}{\lceil n/2\rceil}$. Hence the ith applicant has incentive of playing a different strategy.

Although we did not explicitly compare Π_β and an optimal policy which starts at time $n-1$, it can be shown that the latter is dominated by the former in both cases of n being even or odd (its probability of selecting the best applicant is $\frac{n-1-\lceil n/2\rceil}{n} + \frac{1}{n}\frac{\lceil n/2\rceil}{n-1}$). We ignore this policy because it will choose the ith applicant with probability 0 for $1 \leq i \leq \lceil \frac{n}{2} \rceil$. A similar argument applies for an optimal policy that starts at time n.

$\square$

We have shown how each applicant in slots between 1 and $\frac{n}{2}$ does not want to shorten his window of availability given all others play the strategy $\vec{t}_{-i}$. In the next lemma, we shall show the complement fact: that they also do not want to lengthen their windows of availability.

Lemma 3.8. *Consider an arbitrary applicant i, where $1 \leq i \leq \frac{n}{2}$, and suppose he changes his window of availability to end after time $\frac{n}{2}$. In other words, his new t'_i is such that $i + t'_i \geq \frac{n}{2} + 1$. If all other applicants adhere to the strategy $\vec{t}_{-i}$, then applicant i will get selected with probability strictly less than $\frac{2}{n}$.*

Proof. Denote $\tilde{t} = i + t'_i$, so that $\tilde{t}$ is the last time applicant i is available for hire. Observe that the employer will only make her hiring decisions starting at one of the time epochs $\frac{n}{2}$, $\tilde{t}$, $n-1$, or n. Let Π_α be the policy where the employer hires the best so far starting at time $\tilde{t}$. Also let Π_β be the policy where the employer hires the best so far starting at time $\frac{n}{2}$. We first show that the employer will choose applicants with the policy Π_β. Consider the following three scenarios:

- ith applicant is best at time $\frac{n}{2}$, but is not best at time $\tilde{t}$. When $\tilde{t} \leq n-1$, both Π_α and Π_β select the best applicant with probability $\frac{n-1}{n}$. When $\tilde{t} = n$ and the ith applicant is also best at time $n-1$, then both policies select the best applicant with probability 1. When $\tilde{t} = n$ and the ith applicant is not the best at time $n-1$, then Π_α selects the best with probability $\frac{1}{n}$, and Π_β selects the best with probability $\frac{n-1}{n}$. As such, Π_β strongly dominates Π_α in this scenario.

- ith applicant is best at time $\frac{n}{2}$, and is best at time $\tilde{t}$. The both policies Π_α and Π_β select the best applicant with probability $\frac{\tilde{t}}{n}$ for all $\frac{n}{2} + 1 \leq \tilde{t} \leq n$.

- ith applicant is not best at time $\frac{n}{2}$. Then Π_α selects the best applicant with probability **strictly less than** $\frac{1}{2}$. Π_β, on the other hand, selects the best applicant with probability $\frac{1}{2}$. So that Π_β again strongly dominates Π_α in this scenario.

It follows that the employer will strongly prefer policy Π_β over Π_α. We also observe that an optimal policy which begins at time $n-1$ is dominated by policy Π_β (in the second and third scenarios, it selects the best applicant with probability less than $\frac{1}{2}$). Furthermore, one which begins at time n is also dominated by Π_β. As such, the employer will use Π_β to select applicants.

Under the policy Π_β, the ith applicant only gets selected in scenario (2), and his probability of getting selected is $\frac{1}{\tilde{t}} < \frac{1}{n/2}$. As such, he does not have incentive to lengthen his window of availability to beyond time $\frac{n}{2}$.

$\square$

These results allowed us to conclude with this theorem.

Theorem 3.9. *With $n \geq 6$ and n being even, then $\vec{t}$ is a pure-strategy equilibrium.*

Proof. For $1 \leq i \leq \frac{n}{2}$, applicant i does not want to change his strategy since his probability of getting selected will be lowered. For $\frac{n}{2} < i \leq n$, applicant i changing his strategy does not matter as his probability of getting selected still remains at 0. As such, $\vec{t}$ is a pure-strategy equilibrium. □

3.6 Pareto Optimality And Other Conjectures

It turns out that our constructed equilibrium strategies have a very desirable property: they are Pareto optimal. This fact is proved below.

Proposition 3.10. *The constructed equilibria are Pareto-optimal (and hence, by definition, payoff-dominant).*

Proof. Let p_i^π denote the probability for which a policy π selects applicant i, observe that $\sum_{i=1}^{n} p_i^\pi \leq 1$. But in our constructed equilibrium, the optimal policy selects each of the first $\lceil \frac{n}{2} \rceil$ with probability $\frac{1}{\lceil n/2 \rceil}$, so that $\sum_{i=1}^{n} p_i^\pi = 1$. As such, if we increase the payoff of one applicant, some other applicant's payoff must decrease, so that our constructed equilibria are at the Pareto frontier. □

There are other properties of the game that we observed, but could not prove. These are listed as conjectures here.

- All equilibria have employer's payoff of at least $\frac{1}{2}$. As such, our constructed pure-strategy equilibria are employer's payoff minimum.
- The above statement also holds for *correlated* equilibria.
- As $n \to \infty$, the maximum employer's payoff goes to $\frac{1}{2}$ in an equilibrium.
- For $n \geq 8$, there exists an equilibrium where all applicants are selected with positive probability.

4 Conclusion

In this paper, we presented two game-theoretic approaches to the classical secretary problem. By allowing applicants the freedom to sequentially determine their windows of availability, the employer can improve her probability of selecting the best candidate and be considerate to all applicants at the same time. That is, everyone who is interviewed will get hired with the same probability, independent of their slot positions. If the employer chooses to solicit every applicant's window of availability before the first interview begins, and makes this a binding commitment, then the resulting simultaneous game is guaranteed to almost always have a pure-strategy equilibrium. The lone exception to this fact is when $n = 4$, where its existence is dependent on how the employer optimally selects applicants.

A Non-existence Of Pure-Strategy Equilibrium

The goal of this section is to show there may not exist a pure-strategy equilibrium for applicants. Observe that when there are $n = 4$ applicants, these are the possible corresponding probabilities for getting selected. In the case applicants play $(1, 0, 1, 0)$, the employer selects to optimally hire at time 4. Stepping through all strategies show that none of the below 24 constitute an equilibrium. Note that had the employer optimally selected at time 2 (for the same $(1, 0, 1, 0)$), then there would have been a pure Nash equilibrium, namely $(1, 0, 0, 0)$.

1. $\vec{t} = (0, 0, 0, 0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{11}{24}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	$\frac{1}{2}$	·	·
3	0	0	$\frac{1}{6}$	·
4	0	0	0	$\frac{1}{12}$
$P_i^{\pi(\vec{t})}$	0	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$

2. $\vec{t} = (0, 0, 1, 0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{1}{2}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	0	·	·
3	0	0	0	·
4	0	0	$\frac{1}{4}$	$\frac{1}{4}$
$P_i^{\pi(\vec{t})}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$

3. $\vec{t} = (0, 1, 0, 0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{7}{12}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	0	·	·
3	0	$\frac{1}{3}$	$\frac{1}{3}$	·
4	0	0	0	$\frac{1}{12}$
$P_i^{\pi(\vec{t})}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{12}$

4. $\vec{t} = (0, 1, 1, 0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{2}{3}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	0	·	·
3	0	$\frac{1}{3}$	0	·
4	0	0	$\frac{1}{4}$	$\frac{1}{6}$
$P_i^{\pi(\vec{t})}$	0	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$

5. $\vec{t} = (0, 2, 0, 0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{2}{3}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	0	·	·
3	0	0	$\frac{1}{3}$	·
4	0	$\frac{1}{4}$	0	$\frac{1}{6}$
$P_i^{\pi(\vec{t})}$	0	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$

6. $\vec{t} = (0,2,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{3}{4}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	0	·	·
3	0	0	0	·
4	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$P_i^{\pi(\vec{t})}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

7. $\vec{t} = (1,0,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{1}{2}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	$\frac{1}{2}$	$\frac{1}{2}$	·	·
3	0	0	0	·
4	0	0	0	0
$P_i^{\pi(\vec{t})}$	$\frac{1}{2}$	$\frac{1}{2}$	0	0

8. $\vec{t} = (1,0,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{1}{2}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	0	·	·
3	0	0	0	·
4	0	0	$\frac{1}{4}$	$\frac{1}{4}$
$P_i^{\pi(\vec{t})}$	0	0	$\frac{1}{4}$	$\frac{1}{4}$

9. $\vec{t} = (1,1,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{5}{8}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	$\frac{1}{2}$	0	·	·
3	0	$\frac{1}{3}$	$\frac{1}{6}$	·
4	0	0	0	0
$P_i^{\pi(\vec{t})}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	0

10. $\vec{t} = (1,1,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{2}{3}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	0	·	·
3	0	$\frac{1}{3}$	0	·
4	0	0	$\frac{1}{4}$	$\frac{1}{6}$
$P_i^{\pi(\vec{t})}$	0	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$

11. $\vec{t} = (1,2,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{17}{24}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	$\frac{1}{2}$	0	·	·
3	0	0	$\frac{1}{6}$	·
4	0	$\frac{1}{4}$	0	$\frac{1}{12}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$

12. $\vec{t} = (1,2,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{3}{4}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	0	·	·
3	0	0	0	·
4	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$P_i^{\pi(\vec{t})}$	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

13. $\vec{t} = (2,0,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{5}{8}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	$\frac{1}{2}$	·	·
3	$\frac{1}{3}$	0	$\frac{1}{6}$	·
4	0	0	0	0
$P_i^{\pi(\vec{t})}$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$	0

14. $\vec{t} = (2,0,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{2}{3}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	0	·	·
3	$\frac{1}{3}$	0	0	·
4	0	0	$\frac{1}{4}$	$\frac{1}{6}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{3}$	0	$\frac{1}{4}$	$\frac{1}{6}$

15. $\vec{t} = (2,1,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{3}{4}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	·	·	·
2	0	0	·	·
3	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	·
4	0	0	0	0
$P_i^{\pi(\vec{t})}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	0

16. $\vec{t} = (2,1,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{5}{6}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	$\cdot$	$\cdot$	$\cdot$
2	0	0	$\cdot$	$\cdot$
3	$\frac{1}{3}$	$\frac{1}{3}$	0	$\cdot$
4	0	0	$\frac{1}{4}$	$\frac{1}{12}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{12}$

17. $\vec{t} = (2,2,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{5}{6}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	$\cdot$	$\cdot$	$\cdot$
2	0	0	$\cdot$	$\cdot$
3	$\frac{1}{3}$	0	$\frac{1}{3}$	$\cdot$
4	0	$\frac{1}{4}$	0	$\frac{1}{12}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$

18. $\vec{t} = (2,2,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{11}{12}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	$\cdot$	$\cdot$	$\cdot$
2	0	0	$\cdot$	$\cdot$
3	$\frac{1}{3}$	0	0	$\cdot$
4	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$

19. $\vec{t} = (3,0,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{17}{24}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	$\cdot$	$\cdot$	$\cdot$
2	0	$\frac{1}{2}$	$\cdot$	$\cdot$
3	0	0	$\frac{1}{6}$	$\cdot$
4	$\frac{1}{4}$	0	0	$\frac{1}{12}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$

20. $\vec{t} = (3,0,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{3}{4}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	$\cdot$	$\cdot$	$\cdot$
2	0	0	$\cdot$	$\cdot$
3	0	0	0	$\cdot$
4	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$

21. $\vec{t} = (3,1,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{5}{6}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	$\cdot$	$\cdot$	$\cdot$
2	0	0	$\cdot$	$\cdot$
3	0	$\frac{1}{3}$	$\frac{1}{3}$	$\cdot$
4	$\frac{1}{4}$	0	0	$\frac{1}{12}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{12}$

22. $\vec{t} = (3,1,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{11}{12}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	$\cdot$	$\cdot$	$\cdot$
2	0	0	$\cdot$	$\cdot$
3	0	$\frac{1}{3}$	0	$\cdot$
4	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{6}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$

23. $\vec{t} = (3,2,0,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{11}{12}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	$\cdot$	$\cdot$	$\cdot$
2	0	0	$\cdot$	$\cdot$
3	0	0	$\frac{1}{3}$	$\cdot$
4	$\frac{1}{4}$	$\frac{1}{4}$	0	$\frac{1}{6}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$

24. $\vec{t} = (3,2,1,0)$: Solving yields the employer's probability of selecting the best applicant is $\frac{12}{12}$, and the applicants' probabilities for getting selected are:

$p_{s,i}$	1	2	3	4
1	0	$\cdot$	$\cdot$	$\cdot$
2	0	0	$\cdot$	$\cdot$
3	0	0	0	$\cdot$
4	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
$P_i^{\pi(\vec{t})}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$