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A PROBABILISTIC APPROACH TO IDENTIFYING POSITIVE VALUE CASH FLOWS

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Abstract

We use a probabilistic argument to find sufficient conditions on a cash flow sequence $a_0, a_1, \ldots, a_n$ to ensure that its present value is positive for any nonincreasing sequence of nonnegative interest rates.

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1. Varying interest rates

Consider a cash flow sequence $\boldsymbol{a} = (a_0, a_1, \ldots, a_n)$, where a_i is the amount to be received at the end of period i. (Receiving $a_i < 0$ means making a payment of $-a_i$.) If $r_i \geq 0$ is the interest rate for period i, $i = 1, \ldots, n$, then the present value of this sequence is

$$P(\boldsymbol{a}) = \sum_{i=0}^{n} a_i h(i),$$

where

$$h(0) = 1, \qquad h(i) = \prod_{j=1}^{i} (1 + r_j)^{-1}, \quad i > 0.$$

The necessary and sufficient condition for $P(\boldsymbol{a})$ to be positive for any sequence of nonnegative interest rates is easily obtained.

Theorem 1. *The value* $P(\boldsymbol{a}) > 0$ *for any sequence of nonnegative interest rates if and only if*

$$\sum_{i=0}^{j} a_i \geq 0, \quad \textit{for all } j = 0, \ldots, n-1,$$

and

$$\sum_{i=0}^{n} a_i > 0.$$

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Proof. Sufficiency is proven by using mathematical induction to show that, for all $j = 0, \ldots, n$,

$$\sum_{i=0}^{j} a_i h(i) \geq (a_0 + \cdots + a_j) h(j).$$

As the preceding holds when $j = 0$, assume that it holds for a given j. Then

$$\begin{aligned}\sum_{i=0}^{j+1} a_i h(i) &\geq h(j) \sum_{i=0}^{j} a_i + a_{j+1} h(j+1) \\ &\geq h(j+1) \sum_{i=0}^{j+1} a_i ,\end{aligned}$$

where the last inequality used the sufficient condition along with the fact that $h(j) \geq h(j+1)$. Since $h(n) > 0$, sufficiency follows from the induction hypothesis when $j = n$. To prove necessity, suppose that $\sum_{i=0}^{j} a_i < 0$. Then letting $r_i = 0$ when $i \leq j$, and letting r_i be sufficiently large when $i > j$ produces a negative present value. Also, if $\sum_{i=0}^{n} a_i \leq 0$, then the sequence of interest rates $r_i = 0$ produces a nonpositive present value.

It follows from the proof of necessity that the condition of Theorem 1 is also the necessary and sufficient condition for $P(\boldsymbol{a})$ to be positive for any nondecreasing sequence of nonnegative interest rates. We now give a sufficient condition for when the interest rates are nonincreasing.

Theorem 2. *If the following conditions hold:*

1. $\sum_{i=0}^{n} a_i > 0$;
2. $\sum_{j=0}^{k} (k - j + 1) a_j \geq 0$ *for all* $k = 0, \ldots, n$;

then $P(\boldsymbol{a}) > 0$ *for any nonincreasing sequence of nonnegative interest rates.*

Proof. Let $b_i,\ c_i,\ i = 0, \ldots, n-1$, be nonnegative numbers such that

$$a_i = c_i - b_i, \qquad i = 0, \ldots, n,$$

and let $B = \sum_{i=0}^{n} b_i$. Also, define N as follows:

$$N = \min \left\{ k : \sum_{i=0}^{k} c_i \geq B \right\}.$$

Note that, by condition 1, $\sum_i c_i > B$, and so N is well defined. Now, let

$$d_j = \begin{cases} c_j, & \text{if } j < N, \\ B - \sum_{i=0}^{N-1} c_i, & \text{if } j = N, \\ 0, & \text{if } N < j \leq n. \end{cases}$$

In addition, let X and Y be random variables such that

$$P\{X = j\} = \frac{d_j}{B}, \qquad j = 0, \ldots, n,$$
$$P\{Y = j\} = \frac{b_j}{B}, \qquad j = 0, \ldots, n,$$

and note that

$$E[h(X)] = \sum_{i=0}^{n} \frac{d_i h(i)}{B}, \qquad E[h(Y)] = \sum_{i=0}^{n} \frac{b_i h(i)}{B}.$$

Now, when $0 \leq r_{i+1} \leq r_i$ it is easy to check that the function $f(i) = h(i)$ is a nonincreasing, convex function of i, and a well-known sufficient condition for

$$E[f(X)] \geq E[f(Y)] \quad \text{for all nonincreasing, convex } f$$

is (see Equation (3.A.7) of [4]) that for all k

$$\sum_{i=0}^{k} P\{X \leq i\} \geq \sum_{i=0}^{k} P\{Y \leq i\}. \tag{1}$$

However,

$$\sum_{i=0}^{k} P\{X \leq i\} = \sum_{i=0}^{k} \sum_{j=0}^{i} \frac{d_j}{B} = \sum_{j=0}^{k} (k - j + 1) \frac{d_j}{B}$$

and, similarly,

$$\sum_{i=0}^{k} P\{Y \leq i\} = \sum_{j=0}^{k} (k - j + 1) \frac{b_j}{B}.$$

For $j < N$, $d_j = c_j$, and so when $k < N$ we have by condition 2 of Theorem 2 that

$$\sum_{j=0}^{k} (k - j + 1) d_j \geq \sum_{j=0}^{k} (k - j + 1) b_j.$$

Hence, (1) holds when $k < N$, which, since $P\{X \leq i\} = 1$ when $i \geq N$, implies that (1) also holds for $k \geq N$. Therefore,

$$\sum_{i=0}^{n} d_i h(i) \geq \sum_{i=0}^{n} b_i h(i).$$

Since $c_i \geq d_i$ and $\sum_{i=0}^{n} c_i > \sum_{i=0}^{n} d_i$, the preceding equation yields that

$$\sum_{i=0}^{n} c_i h(i) > \sum_{i=0}^{n} d_i h(i) \geq \sum_{i=0}^{n} b_i h(i),$$

which proves the result.

2. Constant interest rates

When adapted to the case of a constant interest rate r, Theorem 2 gives a sufficient condition for

$$\sum_{i=0}^{n} a_i \alpha^i > 0 \quad \text{for all } 0 < \alpha \le 1\big(\alpha = 1/(1+r)\big).$$

Cash flow streams that have a positive present value for all constant interest rates are of particular interest in finance theory, since it can be demonstrated that such streams can be exploited to achieve arbitrage (see [1] and [2]). In the following we present an easily computed sequence of increasingly weaker sufficient conditions for the polynomial of present value of a given cash flow to be positive for all nonnegative interest rates. While these conditions can be inferred from more general results concerning bounds on the number of positive roots of polynomials [3], we present here a substantially simpler proof.

To begin, we introduce the following notation:

$$P_{\boldsymbol{a}}(\alpha) = \sum_{i=0}^{n} a_i \alpha^i$$

and $s^k = (s_0^k, s_1^k, \ldots, s_n^k)$, where

$$s_i^k = \begin{cases} a_i, & \text{if } k = 0, \\ \sum_{j=0}^{i} s_j^{k-1}, & \text{if } k = 1, 2, \ldots, n. \end{cases}$$

We need the following lemma.

Lemma 1. *We have*

$$P_{s^k}(\alpha) = (1-\alpha) P_{s^{k+1}}(\alpha) + \alpha^{n+1} s_n^k .$$

Proof. To prove this note that for any $\boldsymbol{b} = (b_0, b_1, \ldots, b_n)$,

$$\begin{aligned}(1-\alpha)[b_0 + (b_0 + b_1)\alpha + (b_0 + b_1 + b_2)\alpha^2 + \cdots + (b_0 + \cdots + b_n)\alpha^n] \\ = b_0(1-\alpha^{n+1}) + b_1(\alpha - \alpha^{n+1}) + \cdots + b_n(\alpha^n - \alpha^{n+1}) \\ = P_{\boldsymbol{b}}(\alpha) - \alpha^{n+1}(b_0 + \cdots + b_n),\end{aligned}$$

which completes the proof.

Theorem 3. *If for some nonnegative m the following conditions hold:*

1. $a_0 > 0$;
2. $s_i^m \ge 0$ *for all* $i = 0, \ldots, n$;
3. $s_n^k \ge 0$ *for all* $k = 0, \ldots, m$;

then

$$P_{\boldsymbol{a}}(\alpha) > 0 \quad \textit{for all } 0 \le \alpha \le 1 .$$

Proof. We use backward mathematical induction to show that, for all $k = m, \ldots, 0$

$$P_{s^k}(\alpha) > 0 \quad \text{for all } 0 \le \alpha \le 1.$$

As by conditions 1 and 2 the preceding holds when $k = m$, assume that it holds for $k + 1$. But this implies, by Lemma 1 and condition 3, that

$$P_{s^k}(\alpha) > 0.$$

Thus, by induction the inequality is true for $k = 0$, proving the result.

Remarks. (i) The sufficient condition of Theorem 3 obtained when $m = 2$ is exactly the sufficient condition provided by Theorem 2.

(ii) Other results on determining the number of positive real roots of a polynomial have appeared in [5].

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