The Linear Complementarity Problem, Lemke Algorithm, Perturbation, and the Complexity Class PPAD

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Abstract

We present a single sufficient condition for the processability of the Lemke algorithm for semimonotone Linear Complementarity problems (LCP) which unifies several sufficient conditions for a number of well known subclasses of semimonotone LCPs. In particular, we study the close relationship of these problems to the complexity class PPAD. Next, we show that these classes of LCPs can be reduced (polynomially), by perturbation, to stricter sub-classes and establish an alternative (non-combinatorial) scheme to prove their membership in PPAD. We then identify several of these subclasses as PPAD-complete and discuss the likelihood that the other subclasses (all of which are reducible to LCP's with P matrices) are PPAD-complete. Finally, we present an hierarchy of subclasses of the semimonotone LCP's within the complexity classes of P, PPAD and NP-complete.

1 Introduction

We consider the following linear complementarity problem.

$$
\begin{array}{llrcl}
\mathbf{LCP}(q,M): & \text{find} & z & & \\
 & \text{s.t.} & Mz+q & \geq & 0 \\
 & & z & \geq & 0 \\
 & & z^T(Mz+q) & = & 0
\end{array}
$$

$$\text{where } M \in \boldsymbol{\mathcal{R}}^{\boldsymbol{n\times n}}, \text{ and } z,q \in \boldsymbol{\mathcal{R}}^{\boldsymbol{n\times 1}}$$

We denote by $\mathbf{SOL}\,(q,M)$ the set of all z that solves $\mathbf{LCP}(q,M)$.

Extensive coverage of linear complementarity problem (LCP) theory is available in the monographs [4], [14], [9] and the research articles in their reference lists. One of the interesting aspects of LCP is its range of applications, from well understood and relatively easy problems such as linear and convex quadratic programming problems to $\mathcal{NP}$-hard problems. A major effort in LCP theory had been the study of variants of Lemke algorithm, a simplex-like vertex following algorithm. However, while the method can be applied (and guaranteed to terminate in finite number of iterations) to any nondegenerate LCP, it may terminate without a solution. In this case, the result is meaningful only if at termination the algorithm generates a (polynomial size) certificate that $\mathbf{SOL}\,(q,M) = \emptyset$. Since in general it is not known if such a certificate exists, typically the known cases for which the algorithm works is restricted to problems $\mathbf{LCP}(q,M)$ in which M belongs to the class $\mathbf{Q_0}$, the class of all matrices for which a solution to the the linear part of $\mathbf{LCP}(q, M)$ (that is the first two inequalities sets) implies that $\mathbf{SOL}\,(q, M) \neq \emptyset$. Problems for which it was shown that variants of the Lemke algorithm work were believed to be not as hard as the general problem even when no polynomial upper bound for the method was known to exist. In fact, for most classes, instances with exponential number of steps had been discovered.

The introduction of the $\mathcal{PPAD}$ complexity class in [15] provided a framework for analyzing the complexity of the Lemke algorithm for classes for which it works. This development is significant with respect to LCP theory since $\mathcal{PPAD}$ is believed to be somewhere between the complexity classes $\mathcal{P}$ and $\mathcal{NP}$-complete, lending support to the long standing informal belief that LCPs processable by the Lemke algorithm are 'easier' than the general problem. We discuss this in some details in sections 2 and 3. In section 4 we discuss reduction of $\mathbf{LCP}(q, M)$ via small positive perturbation along the main diagonal of M. Restricting our discussion to the class $\mathbf{E_0}$ of semimonotone matrices that belong to $\mathbf{Q_0}$, we show how it is possible to reduce some known $\mathcal{PPAD}$ LCP problems to three specific cases which are properly contained within the original problems. We also observe in this section the close relationship between perturbation and membership in $\mathcal{PPAD}$. What makes the class $\mathcal{PPAD}$ particularly interesting is the fact that several well known problems such as the problem of finding a Brouwer fixed-point were identified in [15] as $\mathcal{PPAD}$-complete. The recent discoveries in a string of papers ([5], [6], [2] and [3]) that culminated in proving that finding a Nash equilibrium of bimatrix game, is $\mathcal{PPAD}$-complete, has significant consequences in the context of LCP theory as we discuss in sections 5. Finally, we summarize our results in section 6 and discuss various properties of LCP with matrices in ceratin classes as related to their complexity status. We also offer two conjectures which are motivated by the discussion.

Several classes of matrices are discussed throughout the paper. Their definitions can be found in Appendix A.

2 The Complexity Class $\mathcal{PPAD}$

The class $\mathcal{PPAD}$ (Polynomial-time Parity Argument Directed), which was introduced in [15], is a class of problems which can be presented as follows:

Given a directed graph with every node having in-degree and out-degree at most one described by a polynomial-time computable function $f(v)$ that outputs the predecessor and successor of a node v , and a node s with a successor but no predecessors, find a node $t \neq s$ such that it either has no successors or it has no predecessors.

Many important problems, such as the Brouwer fixed-point problem, the search versions of Smiths theorem, the Borsuk-Ulam theorem, as well as Nash equilibrium of bimatrix game, belong to this class (see [15]). Interestingly, the problems in $\mathcal{PPAD}$ are generally believed not to be $\mathcal{NP}$-hard since it had been shown (see [11]) that if there exists a $\mathcal{PPAD}$ problem which is $\mathcal{NP}$-hard, then $\mathcal{NP} = \mathcal{C}o\mathcal{NP}$. What makes the study of this class attractive is that it has been shown that several problems within the class (such as the Brouwer fixed-point problem) is $\mathcal{PPAD}$-complete, with strong circumstantial evidence that this problem is not likely to have a polynomial time algorithm [8]. As we shall see in the next section, several classes of LCP problems can be shown to belong to $\mathcal{PPAD}$ via the Lemke method. The recent proof in [3] that one particular LCP problem, namely the Nash equilibrium of bimatrix game, is $\mathcal{PPAD}$-complete is very significant to the theory of LCP as we shall discuss later.

3 The Lemke Algorithm

In this section we consider the Lemke algorithm - Scheme I as applied to nondegenerate LCP problems (see [4], 4.4.5) which we simply refer to as the Lemke algorithm. We shall assume throughout the paper that the problems at hand are nondegenerate and that their coefficient matrices belong to $\mathbf{Q_0}$. For the definition of a nondegeneracy and the justification for assuming it we refer the reader to Appendix B. Given an $\mathbf{LCP}(q, M)$ with $q \notin \mathbf{R}^n_+$* and a covering vector $d \in \mathbf{R}^n_{++}$, the Lemke Algorithm generates (in finite number of pivots) either a solution to $\mathbf{LCP}(q, M)$ or a so-called secondary ray. Specifically, let

$$\begin{aligned}\mathbf{SR}(q, M) \;=\; \{\; & z, u, d \in \mathbf{R}^n_+ \;,\; z_0 > 0 \;,\; u_o \geq 0 \;\mid\; d_i = 0 \Rightarrow u_i = 0 \; (i = 1, \ldots, n) \;,\; z \in \mathbf{SOL}\,(q + dz_0, M) \;, \\ & u \in \mathbf{SOL}\,(u_0 d, M) - \{0\} \;,\; z^T(Mu + u_0 d) = 0 \;,\; u^T(Mz + q + z_0 d) = 0 \;\}\end{aligned}$$

then, the Lemke algorithm generates either $\bar{z} \in \mathbf{SOL}\,(q, M)$ or $(\bar{z}, \bar{u}, \bar{d}, \bar{z}_o, \bar{u}_o) \in \mathbf{SR}(q, M)$.

In the following we shall focus on the following problem:

$\mathbf{Y} - \mathbf{LCP}(q, M)$: *Given that* $M \in \mathbf{Y}$, *either find* $z \in \mathbf{SOL}\,(q, M)$ *or show that* $\mathbf{SOL}\,(q, M) = \emptyset$

As is demonstrated in [4] and [14], the two major monographs about LCP, a great deal of effort has been devoted to identify classes of matrices $\mathbf{Y}$ for which the Lemke algorithm is guaranteed to solve $\mathbf{Y} - \mathbf{LCP}(q, M)$. In particular, a number of matrices' classes $\mathbf{Y}$ had been shown to have the following property:

SRP (*Secondary Ray Property*) : $\mathbf{SR}(q, M) \neq \emptyset$ *implies that* $\mathbf{SOL}\,(q, M) = \emptyset$

a property that obviously allows problem $\mathbf{Y} - \mathbf{LCP}(q, M)$ to be solved by the Lemke algorithm. We say, in this case, that $\mathbf{Y}$ is *Lemke processable.* The properties of the Lemke algorithm as discussed in [4], and in particular, the intrinsic orientation of the algorithm as presented in [17], verify that the path corresponding to the the Lemke algorithm on the graph induced by the vertices of the polyhedral set determined by the linear inequalities of the LCP, satisfies the requirements specified in the definition of the class $\mathcal{PPAD}$. This leads to

Theorem 3.1 *Any problem* $\mathbf{Y} - \mathbf{LCP}(q, M)$ *for which* $\mathbf{Y}$ *is Lemke processable, belongs to* $\mathcal{PPAD}$.

References [4] and [14] describe in detail several of the classes that have been shown to be Lemke processable. These classes, among others, include $\mathbf{P}, \mathbf{RSU}, \mathbf{CSU}, \mathbf{SU}, \mathbf{P_0} \cap \mathbf{Q_0}, \mathbf{C_*}, \mathbf{C_+}, \mathbf{C}, \mathbf{E}$ and $\mathbf{L}$ (the definition of these classes can be found in Appendix A). All the preceding classes belong to the class $\mathbf{E_0}$ of semimonotone matrices. This class is particulary suited for the Lemke algorithm because of the following property: whenever $M \in \mathbf{E_0}$, $u_0 = 0$ in any secondary ray generated by the Lemke algorithm (see [4], 4.4.11), so $\mathbf{SR}(q, M)$ is specialized in this case to:

$$\begin{aligned}\mathbf{SRE_0}(q, M) \;=\; \{\; & z, u, d \in \mathbf{R}^n_+ \;,\; z_0 > 0 \;\mid\; d_i = 0 \Rightarrow u_i = 0 \; (i = 1, \ldots, n) \;,\; z \in \mathbf{SOL}\,(q + dz_0, M) \;, \\ & u \in \mathbf{SOL}\,(0, M) - \{0\} \;,\; z^T M u = 0 \;,\; u^T(Mz + q + z_0 d) = 0 \;\}\end{aligned}$$

In fact, for all the classes $\mathbf{Y} \subseteq \mathbf{E_0}$ for which $\mathbf{Y}$ is Lemke processable, the key observation is that a member of $\mathbf{SRE_0}(q, M)$ can be easily converted to a certificate for $\mathbf{SOL}\,(q, M) = \emptyset$.

Interestingly, it had been informally believed by the LCP research community that $\mathbf{Y} - \mathbf{LCP}(q, M)$ problems with Lemke processable $\mathbf{Y}$ are 'easier', even though in most cases there was no evidence that the Lemke Algorithm is polynomially bounded. In fact, for most of these classes there exist instances that require an exponential number of steps (see for example, [4], 4.10.4 for LCP with P matrices and [16] for LCP corresponding to bimatrix games). The introduction of the $\mathcal{PPAD}$ class in [15] provided a complexity framework for all $\mathbf{Y} - \mathbf{LCP}$ problems for which $\mathbf{Y}$ is Lemke processable. In particular, it provided a justification to the belief that $\mathbf{Y} - \mathbf{LCP}$ problems with

*Whenever $q \geq 0$, trivially $0 \in \mathbf{SOL}\,(q, M)$. So throughout the paper we assume that $q \notin R^+_n$, and as a consequence, $z \in SOL(q, M) \Rightarrow z \neq 0$.

Lemke processable $\mathbf{Y}$ are not likely to be $\mathcal{NP}$-hard since it had been shown (see [11]) that if there exists a $\mathcal{PPAD}$ problem which is $\mathcal{NP}$ hard, then $\mathcal{NP} = \mathcal{C}o\mathcal{NP}$. In addition, the existence of $\mathcal{PPAD}$-complete problems (introduced first in [15]) demonstrates that all $\mathbf{Y}-\mathbf{LCP}$ problems with Lemke processable $\mathbf{Y}$ can be reduced to any of the $\mathcal{PPAD}$-complete problems. The recent proof in [3] that the problem of finding a Nash equilibrium for bimatrix game is $\mathcal{PPAD}$-complete is significant in the context of LCP theory since it is well-known that this problem can be formulated as a $\mathbf{Y}-\mathbf{LCP}(q, M)$ problem. So, as it turned out, all Lemke processable $\mathbf{Y}-\mathbf{LCP}(q, M)$ can be reduced to the LCP formulation of a bimatrix game, a rather startling development.

4 Reduction of LCP Problems via Perturbation

The idea of perturbing a matrix M by small positive ϵ along the main diagonal prior to solving $\mathbf{LCP}(q, M)$ was discussed in the LCP literature as a possible strategy to convert the problem into a possibly easier one (see the discussion about regularization in [4], 5.6). The key for doing it successfully is the ability to easily convert a solution to the perturbed problem into a solution to the original problem. Over the years it had been shown that for several classes, but by no means all, this strategy is workable. In this section we discuss some sufficient conditions for the perturbation to work and show that it is closely related to membership in the class $\mathcal{PPAD}$.

For $\epsilon > 0$, denote $M(\epsilon) = M + \epsilon I$. We say that an $\mathbf{LCP}(q, M)$ is *reducible by perturbation* to $\mathbf{LCP}(q, M(\epsilon))$, if there exits $\bar{\epsilon} > 0$ such that for all $0 < \epsilon \leq \bar{\epsilon}$

(i) $\mathbf{SOL}\,(q, M(\epsilon)) \neq \emptyset$.

(ii) Given $\bar{z} \in \mathbf{SOL}\,(q, M(\epsilon))$, it is possible to find (in polynomial time) either $z \in \mathbf{SOL}\,(q, M)$ or a certificate for $\mathbf{SOL}\,(q, M) = \emptyset$.

Since it is well known that for $M \in \mathbf{E_0}$ and $\epsilon > 0$, $M(\epsilon) \in \mathbf{E}$, and since for $M \in \mathbf{E}$ and all q, $\mathbf{SOL}\,(q, M) \neq \emptyset$, it is clear that requirement (i) of the reduction is satisfied for the class $\mathbf{E_0}$, the class of semimonotone matrices that has been extensively studied and contains several major subclasses (as well as all of the examples of Lemke processable mentioned following Theorem 3.1).

In the following we shall show that a solution to $\mathbf{LCP}(q, M(\epsilon))$ with $M \in \mathbf{E_0}$ either provides (in polynomial time) a solution to $\mathbf{LCP}(q, M)$ or a member in $\mathbf{SRE_0}(q, M)$. Thus, we shall establish that all $\mathbf{Y}-\mathbf{LCP}(q, M)$ with Lemke processable $\mathbf{Y}$ in $\mathbf{E_0}$ are reducible by perturbation to $\mathbf{LCP}(q, M(\epsilon))$.

Let $\bar{z} \in \mathbf{SOL}\,(q, M(\epsilon))$, $\alpha = \{j \mid \bar{z}_j > 0\}$ and $\bar{\alpha}$ the complement of α with respect to $\{1, 2, \ldots, n\}$. Denote

$$C_\alpha(\epsilon) = \begin{pmatrix} -M_{\alpha\alpha}(\epsilon) & 0 \\ -M_{\bar{\alpha}\alpha} & I \end{pmatrix} \qquad \text{where } C_\alpha \equiv C_\alpha(0)$$

Define the following sets:

$$\begin{aligned} S_\alpha(\epsilon) &= \{x \geq 0 \mid C_\alpha(\epsilon)x = q\} \\ S_\alpha &= \{x \geq 0 \mid C_\alpha x = q\} \\ T_\alpha &= \{0 \neq y \geq 0 \mid C_\alpha y = 0\} \end{aligned}$$

Lemma 4.1 *Suppose that $T_\alpha = \emptyset$. Then, for sufficiently small $\epsilon > 0$, $S_\alpha(\epsilon) \neq \emptyset$ implies $S_\alpha \neq \emptyset$.*

Proof

Suppose $T_\alpha = \emptyset$, then, by Gordan's theorem (see [4], 2.7.10), there exists $\bar{v}$ such that $\bar{v}^T C_\alpha(\epsilon) < 0$. Now, suppose $S_\alpha = \emptyset$, then, by Farkas' lemma (see [4], 2.7.7.), there exists $\bar{x}$ such that $\bar{x}^T C_\alpha \leq 0$ and $\bar{x}^T q > 0$. Let $\theta > 0$ be sufficiently large number such that $\bar{p} \equiv \bar{v} + \theta \bar{x}$ satisfies $\bar{p}^T q > 0$. Writing (rearranging rows and columns as necessary) $\bar{p}^T = (\bar{p}_\alpha^T \ , \ \bar{p}_{\bar{\alpha}}^T)$, we get

$$\bar{p}^T C_\alpha(\epsilon) = (\bar{p}_\alpha^T \ , \ \bar{p}_{\bar{\alpha}}^T) \begin{pmatrix} -M_{\alpha\alpha} - \epsilon I & 0 \\ -M_{\bar{\alpha}\alpha} & I \end{pmatrix} = \bar{p}^T C_\alpha - \epsilon(\bar{p}_\alpha^T, 0)$$

Considering the preceding and noting that $\bar{p}^T C_\alpha < 0$, we get that for a sufficiently small ϵ, $\bar{p}^T C_\alpha(\epsilon) \leq 0$. Thus, since $\bar{p}^T q > 0$, we conclude, by Farka's lemma, that for sufficiently small ϵ, $S_\alpha(\epsilon) = \emptyset$. ■

Lemma 4.2 *Suppose that $\bar{y} \in T_\alpha$ and let $\bar{u}$ be defined such that $\bar{u}_i = \bar{y}_i$ for $i \in \alpha$ and $\bar{u}_i = 0$ for $i \in \bar{\alpha}$. Then, $(\bar{z}, \bar{u}, \bar{z}, \epsilon) \in \mathbf{SRE_0}(q, M)$.*

Proof

Given the definitions of $\bar{z}$ and $\bar{u}$ and since $\epsilon > 0$, the claim can be easily verified. ■

Thus, given $\bar{z} \in \mathbf{SOL}\,(q, M(\epsilon))$ (for sufficiently small ϵ), one can check whether $T_\alpha = \emptyset$. If it is, then by Lemma 4.1, $S_\alpha \neq \emptyset$. So finding $\bar{x}$ in S_α will yield a solution to $\mathbf{LCP}(q, M)$. On the other hand, if $T_\alpha \neq \emptyset$, then a $\bar{u}$ together with $\bar{z}$ and ϵ, as prescribed by Lemma 4.2, provides a member of $\mathbf{SRE_0}(q, M)$. Note that any of the tasks above can be performed by a polynomial time LP solver[1]. Considering the preceding together with the discussion in section 3 concerning problem $\mathbf{Y} - \mathbf{LCP}(q, M)$ and Lemke processable $\mathbf{Y}$, we have

Theorem 4.3 *Suppose a class of matrices $\mathbf{Y} \subseteq \mathbf{E_0}$ is Lemke processable. If $M \in \mathbf{Y}$, then for every q, $\mathbf{LCP}(q, M)$ is reducible by perturbation to $\mathbf{LCP}(q, M(\epsilon))$.*

So far we have established that any problem $\mathbf{Y} - \mathbf{LCP}$ with $\mathbf{Y} \subseteq \mathbf{E_0}$ which is certified by the Lemke Algorithm to be in $\mathcal{PPAD}$, is also reducible by perturbation. We can also show that if the problem $\mathbf{Y} - \mathbf{LCP}$ with $\mathbf{Y} \subseteq \mathbf{E_0}$ is reducible by perturbation then it belongs to $\mathcal{PPAD}$. The key to this is the fact (see [4], 4.4.9) that the vector z in $\mathbf{SRE_0}(q, M)$ certifies that $M \notin \mathbf{E}$. Now consider the problem

$$\mathbf{LME}(q, M) \ : \ \text{Either find a solution to } \mathbf{LCP}(q, M) \text{ or certify that } M \notin \mathbf{E}$$

It should be clear from the preceding discussion that $\mathbf{LME}(q, M)$ can always be solved by the Lemke Algorithm, so it belongs to $\mathcal{PPAD}$. In particular, the Lemke Algorithm when applied to $\mathbf{LCP}(q, M(\epsilon))$ with $M \in \mathbf{E_0}$ provides a solution to the problem because $M(\epsilon) \in \mathbf{E}$ for any $\epsilon > 0$. Now suppose that the class of matrices $\mathbf{Y} \subseteq \mathbf{E_0}$ is reducible by perturbation, then by using the solution to $\mathbf{LCP}(q, M(\epsilon))$ we can find (in polynomial time) either a solution to $\mathbf{SOL}\,(q, M)$ or a certificate for $\mathbf{SOL}\,(q, M) = \emptyset$. Interestingly, this is true regardless of the nature of the proof that the class is reducible by perturbation (so it is not necessary to rely on analysis of the secondary ray). Thus we have

Theorem 4.4 *Suppose a class of matrices $\mathbf{Y} \subseteq \mathbf{E_0}$ is reducible by perturbation. Then $\mathbf{Y} - \mathbf{LCP}$ belongs to $\mathcal{PPAD}$.*

[1] It can be shown by standard arguments that it is possible to identify the required small ϵ while maintaining the size of $M(\epsilon)$ as a polynomial function of the the size of M.

It should be noted that reduction by perturbation is helpful for the complexity analysis of classes of matrices within $\mathbf{E_0}$ beyond verifying membership in $\mathcal{PPAD}$. In fact, there are two important cases in which the reduction transforms a $\mathcal{PPAD}$ class $\mathbf{Y} \subseteq \mathbf{E_0}$ into a proper subclass of $\mathbf{Y}$. The first is the class $\mathbf{P_0}$ of matrices with nonnegative principal minors whose perturbation (see [4], 3.4.2) forms the class $\mathbf{P}$ of matrices with positive principal minors. The classes $\mathbf{CSU}, \mathbf{RSU}, \mathbf{SU},$ and $\mathbf{P_0} \cap \mathbf{Q_0}$ are contained in $\mathbf{P_0}$ while their intersection contains $\mathbf{P}$. Noting the remark following Theorem 4.3, we can conclude that of all these LCP problems belong to $\mathcal{PPAD}$ and the complexity of any of these problems is the same as $\mathbf{P} - \mathbf{LCP}(q, M)$. The other subclass of interest is $\mathbf{C_0}$, the class of copositive matrices whose perturbation (see [4], 3.8.1) forms the class $\mathbf{C}$ of strictly copositive matrices. Following similar arguments as in the preceding discussion (and recalling the remark following Theorem 4.3), we can conclude that the classes $\mathbf{C_+}$ and $\mathbf{C_*}$ which are contained in $\mathbf{C_0}$ while their intersection contains $\mathbf{C}$, belong to $\mathcal{PPAD}$ and the complexity of both LCP problems is the same as $\mathbf{C} - \mathbf{LCP}(q, M)$. Another subclass of $\mathbf{E_0}$ that had been extensively studied is $\mathbf{L}$ which belongs to neither $\mathbf{P_0}$ nor $\mathbf{C_0}$ but is reduced to $\mathbf{E}$ which is a proper subset of $\mathbf{L}$ itself.

5 $\mathcal{PPAD}$-complete LCP problems

In this section we show that several of the matrices' classes that were identified in section 3 as $\mathcal{PPAD}$ are in fact $\mathcal{PPAD}$-complete. Our starting point is the problem of finding a Nash equilibrium for bimatrix game which can be stated as follows (see [4], 1.2) :

Given $A \in \mathcal{R}^{m \times n}$, $B \in \mathcal{R}^{n \times m}$, find $\bar{x}, \bar{y}$ such that

$$\begin{array}{rcll} \bar{x}^T A \bar{y} & \leq & x^T A \bar{y} & \text{for all } x \geq 0 \quad \text{and} \quad e^T x = 1 \\ \bar{x}^T B \bar{y} & \leq & \bar{x}^T B y & \text{for all } y \geq 0 \quad \text{and} \quad e^T y = 1 \end{array}$$

Assuming without loss of generality that $A, B > 0$, the common formulation of the bimatrix game problem as $\mathbf{LCP}(q, M)$ has $q = \begin{pmatrix} -e \\ -e \end{pmatrix}$ and $M = \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix}$. While this LCP problem is not Lemke processable, a small modifications in M and q allow for an equivalent LCP which belong to two of the classes that were identified in the preceding sections to be $\mathcal{PPAD}$. The first one (introduced in [7]), uses $q = \begin{pmatrix} -e \\ e \end{pmatrix}$ and $M = \begin{pmatrix} 0 & A \\ -B^T & 0 \end{pmatrix}$ so $M \in \mathbf{L}$. The second one (introduced in [12]), uses $q = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ and $M = \begin{pmatrix} D & A & -e & 0 \\ B^T & 0 & 0 & -e \\ e^T & 0 & 0 & 0 \\ 0 & e^T & 0 & 0 \end{pmatrix}$ where D is a matrix whose entries are all 1's. In this case, $M \in \mathbf{C_+}$. These formulations place the bimatrix game problem as $\mathcal{PPAD}$ (it was noticed in [15] that bimatrix games belong to $\mathcal{PPAD}$ via the special variant of the Lemke method (see [4], 4.4.21) that was designed to solve the common formulation). Somewhat surprisingly, it was established in [3] that the bimatrix game problem is $\mathcal{PPAD}$-complete. Thus, both $\mathbf{L}-\mathbf{LCP}(q, M)$ and $\mathbf{C_+}-\mathbf{LCP}(q, M)$ are $\mathcal{PPAD}$-complete. In addition, since $\mathbf{C_+} \subset \mathbf{C_*}$ we have that $\mathbf{C_*} - \mathbf{LCP}(q, M)$ is $\mathcal{PPAD}$-complete as well. Now, as we demonstrated in section 4, $\mathbf{L} - \mathbf{LCP}(q, M)$ and $\mathbf{C_+} - \mathbf{LCP}(q, M)$ are reducible via perturbation to $\mathbf{E} - \mathbf{LCP}(q, M)$ and $\mathbf{C} - \mathbf{LCP}(q, M)$ respectively, establishing these two problems as $\mathcal{PPAD}$-complete. Finally, problem $\mathbf{LME}(q, M)$ is clearly $\mathcal{PPAD}$-complete.

6 Discussion

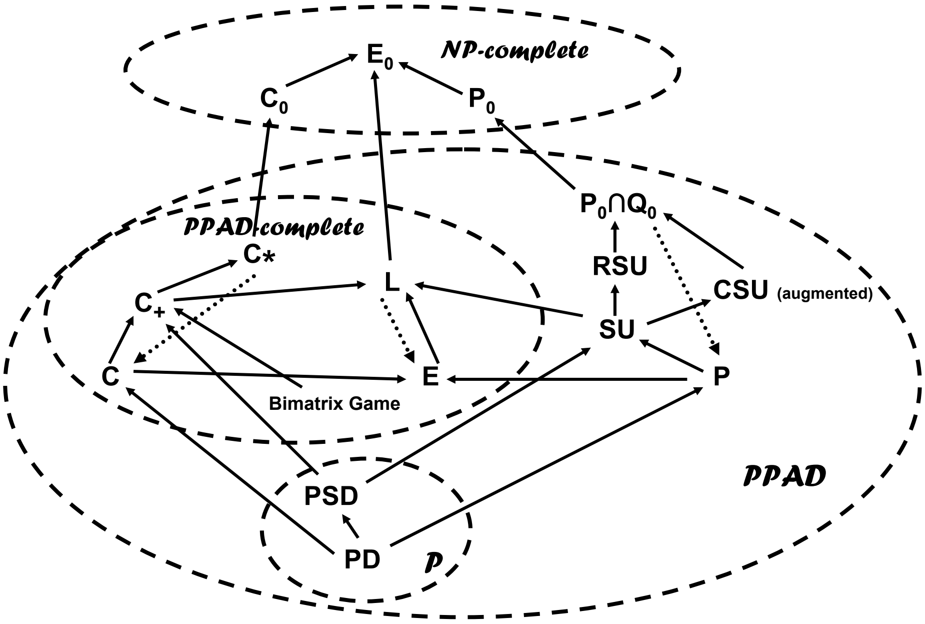

Figure 1

Figure 1 summarizes the relationship among the LCP's classes studied in this paper. A solid arrow form class $\mathbf{X}$ to class $\mathbf{Y}$ indicates that $\mathbf{X} \subset \mathbf{Y}$, a dotted arrow form class $\mathbf{X}$ to class $\mathbf{Y}$ indicates that $\mathbf{X} - \mathbf{LCP}(q, M)$ is reducible by perturbation to $\mathbf{Y} - \mathbf{LCP}(q, M(\epsilon))$.

Note that the classes in Figure 1 can be considered as nodes on a directed graph whose source is the class of matrices $\mathbf{PD}$ and its sink is the class $\mathbf{E_0}$. A directed path along solid edges leads to more general class of matrices (and thus to corresponding harder problems). The easiest problems in the graph, which are known to belong to $\mathcal{P}$, are those related to positive definite and semi-definite matrices. Two well known generalizations of positive semi-definite matrices are copositive matrices ($\mathbf{C_0}$) and matrices with non-negative principles ($\mathbf{P_0}$), where both classes are special cases of semimonotone matrices ($\mathbf{E_0}$). However, it is known that problem $\mathbf{P_0} - \mathbf{LCP}(q, M)$ is $\mathcal{NP}$-complete (see [9]). We show in Appendix C that $\mathbf{C_0} - \mathbf{LCP}(q, M)$ is also $\mathcal{NP}$-complete. In between these two cases lie all the classes corresponding to the $\mathcal{PPAD}$ problems discussed in this paper. Considering reduction via perturbation we can partition all the $\mathcal{PPAD}$ problems in the graph into three groups. All the $\mathcal{PPAD}$ subclasses in $\mathbf{C_0}$ that we discussed in the previous sections (namely $\mathbf{C_+}, \mathbf{C_*}$ and $\mathbf{C}$) are (complexity wise) equivalent to $\mathbf{C}$ which is $\mathcal{PPAD}$-complete. Similarly, $\mathbf{L}$ is equivalent to $\mathbf{E}$ that is $\mathcal{PPAD}$-complete. The $\mathcal{PPAD}$ completeness of a $\mathbf{Y} - \mathbf{LCP}(q, M)$ means that any of the problems in $\mathcal{PPAD}$ can reduced to it, so all the $\mathcal{PPAD}$-complete problems are hard (within the $\mathcal{PPAD}$ class) and generally believed not to be solvable in polynomial time (see, for example, [8]). On the other hand, all the $\mathcal{PPAD}$ subclasses in $\mathbf{P_0}$ that we discussed in the previous sections (namely $\mathbf{P}, \mathbf{CSU}, \mathbf{RSU}, \mathbf{SU},$ and $\mathbf{P_0} \cap \mathbf{Q_0}$) can be reduced via perturbation to $\mathbf{P} - \mathbf{LCP}(q, M)$. It is not known whether this problem is $\mathcal{PPAD}$-complete and it is still an open question whether it can be solved in polynomial time.

Considering some of the known properties of all of the preceding subclasses, we offer the following (speculative) possible explanation to their complexity status. To facilitate the discussion we shall refer, as needed, to the *optimality version* of $\mathbf{LCP}(q, M)$ which is the following problem:

$$\begin{array}{llll} \mathbf{OLCP}(q,M): & \text{minimize} & z^T(Mz+q) & \\ & \text{s.t.} & Mz+q & \geq 0 \\ & & z & \geq 0 \end{array}$$

with the obvious observation that $\bar{z}$ solves $\mathbf{OLCP}(q, M)$ with $\bar{z}^T(M\bar{z} + q) = 0$ if , and only if, $\bar{z} \in \mathbf{SOL}\,(q, M)$.

The key property for the membership of $\mathbf{PSD}-\mathbf{LCP}(q, M)$ in the complexity class $\mathcal{P}$ is that its optimization version is convex, a strong property that often leads to polynomial time algorithms. In particular, $\mathbf{PSD}-\mathbf{LCP}(q, M)$ has the following three 'nice' properties:

(P1) $\{q \mid \mathbf{SOL}\,(q, M) \neq \emptyset\}$ *is convex.*

(P2) $\mathbf{SOL}\,(q, M)$ *is convex.*

(P3) *For each vector q, the Kuhn-Tucker-Karush conditions for* $\mathbf{OLCP}(q, M)$ *are necessary and sufficient for global optimality of the problem.*

It is well known that:

P1 is satisfied if, and only if, $M \in \mathbf{Q_0}$ (see [4], 3.2.1).

P2 is satisfied if, and only if, $M \in \mathbf{CSU}$ (see [4], 3.5.8).

P3 is satisfied if, and only if, $M \in \mathbf{RSU}$ (see [4], 3.5.4).

Our first observation is that for all the problems $\mathbf{Y} - \mathbf{LCP}(q, M)$ that were identified in this paper as $\mathcal{PPAD}$, $\mathbf{Y} \subset \mathbf{Q_0}$[‡] which guarantees the existence of a polynomial size certificate for $\mathbf{SOL}\,(q, M) = \emptyset$ whenever this is the case. We believe that this property is the key for membership in $\mathcal{PPAD}$. So we offer the following conjecture,

Conjecture 1: $(\mathbf{E_0} \cap \mathbf{Q_0}) - \mathbf{LCP}(q, M)$ *belongs to* $\mathcal{PPAD}$.

Recall that according to our results in section 4, all the $\mathcal{PPAD}$ problems in Figure 1 are reducible to either classes $\mathbf{C}, \mathbf{E}$ which are $\mathcal{PPAD}$-complete but do not necessarily satisfy properties P2-P3, or to class $\mathbf{P}$ which is not known to be $\mathcal{PPAD}$-complete but does satisfy properties P2-P3. We believe that this observation points to the likelihood that $\mathbf{P} - \mathbf{LCP}(q, M)$ is 'easier' than $\mathbf{C} - \mathbf{LCP}(q, M)$ or $\mathbf{E} - \mathbf{LCP}(q, M)$ though probably not in the complexity class $\mathcal{P}$, thus the following additional conjecture:

Conjecture 2: $\mathbf{P} - \mathbf{LCP}(q, M)$ *is not* $\mathcal{PPAD}$*-complete.*

With the discovery that the bimatrix game is $\mathcal{PPAD}$-complete, we believe that the complexity classification of the **P** LCP problem becomes one of the major open problem in LCP theory. The fact that there exist instances for which the Lemke algorithm takes exponential number of steps to solve $\mathbf{P} - \mathbf{LCP}(q, M)$, does not by itself contradicts Conjecture 2. In a way, this is somewhat analogous to the case of linear programming where variants of the Simplex method were known to take exponential number of pivots but different algorithms, such as Ellipsoid or Interior Point, had been shown to take polynomial number of steps. Interesting investigations along these lines form an active research area within LCP theory (see for example, [9]).

[‡]While class **CSU** is not in $\mathbf{Q_0}$, the corresponding LCP can be augmented to an equivalent problem whose coefficient matrix is in $\mathbf{Q_0}$ (see [4], 3.7.16).

Finally, we comment on what might seem as a surprising observation that a hard problem like $\mathbf{P_0} - \mathbf{LCP}(q, M)$ becomes what is believed to be much easier problem by an arbitrary small perturbation. In fact, the following simple manipulation shows that all LCP problem can be converted to $\mathbf{P_0} - \mathbf{LCP}(q, M)$ whose perturbed version is trivially solved with a solution that has nothing to do with the original problem.

Given $\mathbf{LCP}(q, M)$, we construct a related problem $\mathbf{LCP}(q', M')$ where

$$q' = \begin{pmatrix} 0 \\ q \\ 0 \end{pmatrix} \quad \text{and} \quad M' = \begin{pmatrix} 0 & I & -I \\ 0 & 0 & M \\ 0 & 0 & 0 \end{pmatrix}$$

Note that M' is a $\mathbf{P_0}$ matrix because it's a triangular matrix with zero diagonal entries. It can be easily verified that if (x, y, z) solves $\mathbf{LCP}(q', M')$, then, z solves $\mathbf{LCP}(q, M)$. On the other hand, if z solves $\mathbf{LCP}(q, M)$, then $(0, z, z)$ solves $\mathbf{LCP}(q', M')$. However, $(x = 0, y = q^+/\epsilon, z = 0)^\dagger$ solves $\mathbf{LCP}(q', M'(\epsilon))$, a solution which is independent of M(!) and thus is meaningless with respect to $\mathbf{LCP}(q, M)$.

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†Where $q_i^+ = \max(q_i, 0)$.

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Appendices

A Classes of Matrices

We will use the following definitions of different subclasses. We have used [4] as the primary reference for these definitions. Suppose matrix $M \in \boldsymbol{\mathcal{R}}^{\boldsymbol{n\times n}}$.

M is a positive semi-definite or a **PSD** matrix if $x^T Mx \geq 0$ for any x.

M is a positive definite or a **PD** matrix if $x^T Mx > 0$ for any $x \neq 0$.

M is a $\mathbf{P_0}$ matrix if all its principal minors are nonnegative.

M is a **P** matrix if all its principal minors are positive.

M is a column-sufficient or a **CSU** matrix if for any vector $x \in \boldsymbol{\mathcal{R}}^{\boldsymbol{n\times 1}}$ and for every $i = 1, 2, \ldots .m$, $x_i(Mx)_i \leq 0$, then $x_i(Mx)_i = 0$ for all $i = 1, 2, \ldots, m$.

M is a row-sufficient or a **RSU** matrix if its transpose is a **CSU** matrix.

M is a **SU** matrix if $M \in \mathbf{RSU} \cap \mathbf{CSU}$.

M is a copositive or a $\mathbf{C_0}$ matrix if $x^T Mx \geq 0$ for any $x \geq 0$.

M is a strictly copositive or a **C** matrix if $x^T Mx > 0$ for any $0 \neq x \geq 0$.

M is a copositive-plus or a $\mathbf{C_+}$ matrix if M is copositive and $x^T Mx = 0,\ x \geq 0$ implies that $(M+M^T)x = 0$.

M is a copositive-star or a $\mathbf{C_*}$ matrix if M is copositive and $x^T Mx = 0,\ Mx \geq 0,\ x \geq 0$ implies that $M^T x \leq 0$.

M is a semimonotone or a $\mathbf{E_0}$ matrix if for any non-zero $x \geq 0$, there exists an index k such that $x_k > 0$ and $(Mx)_k \geq 0$.

M is a strictly semimonotone or a **E** matrix if for any non-zero $x \geq 0$, there exists index k such that $x_k > 0$ and $(Mx)_k > 0$.

M is a $\mathbf{E_1}$ matrix if for every nonzero vector $z \in \mathbf{SOL}\,(0, M)$, there exists non-negative diagonal matrices D_1 and D_2 such that $D_2 z \neq 0$ and $(D_1 M + M^T D_2)z = 0$.

M is a **L** matrix if M is $\mathbf{E_0}$ and $\mathbf{E_1}$.

M is a $\mathbf{Q_0}$ matrix if $\mathbf{SOL}\,(q, M) \neq \emptyset$ for all q whenever there exists a $z \geq 0$ such that $Mz + q \geq 0$.

B LCP with $\mathbf{Q_0}$ Matrices and Non-degeneracy

In this section, we consider $\mathbf{LCP}(q, M)$ with $M \in \mathbf{Q_0}$ and show that we can assume that the problem is non-degenerate from a complexity point of view. Specifically, if x is a solution to $\mathbf{LCP}(q, M)$ then either $x_i > 0$ or $(Mx)_i > 0$ for all i. We show how to calculate a q' of size polynomial in size of (q, M) so that $\mathbf{LCP}(q, M)$ is non-degenerate.

The following result is a direct application of Theorem 10.1 and 10.5 in [13].

Lemma B.1 *Suppose $A \in \mathcal{R}^{m\times n}$ and has a rank m and $b \in \mathcal{R}^{m\times 1}$. Then, there exists a positive number $\bar{\epsilon}(b, A) > 0$, such that whenever $0 < \epsilon < \bar{\epsilon}(b, A)$, all basic feasible solutions to the linear system: $Ax = b + (\epsilon,\ \epsilon^2, ..., \epsilon^m)^T,\ x \geq 0$ are non-degenerate and if B is a feasible basis for $Ax = b + (\epsilon, \epsilon^2, ..., \epsilon^m)^T,\ x \geq 0$ then B is also a feasible basis for $Ax = b,\ x \geq 0$.*

We get the following result when we apply the above lemma to $\mathbf{LCP}(q, M)$.

Corollary B.2 *Suppose $M \in \mathbf{Q_0}$ and $q \in K(M)$. Further suppose that $q'(t) = q + (t, t^2, ..., t^n)^T$. Then, $\mathbf{LCP}(q'(t), M)$ has a solution for all $t > 0$. Moreoever, there exists a $\bar{t}(q, M) > 0$ such that for any positive $t < \bar{t}(q, M)$, if $q'(t) \in$ pos C_α for some index set α where C_α is invertible, then $q'(t) \in$ int pos C_α and $q \in$ pos C_α. Moreover, $\bar{t}(q, M)$ has a size that is polynomial in the size of M and q if they have rational entries.*

C $\mathbf{C_0} - \mathbf{LCP}(q, M)$ is $\mathcal{NP}$-complete

We consider problem $\mathbf{C_0} - \mathbf{LCP}(q, M)$ and show that it is $\mathcal{NP}$-complete by reducing it (motivated by the reduction of $\mathbf{P_0} - \mathbf{LCP}(q, M)$ presented in [9]) to a knapsack problem.

Consider the Knapsack problem: Given a non-negative integers $a_1, a_2, \ldots, a_m, b$, find $w_1, w_2, \ldots, w_m$ such that

$$\sum_{i=1}^{m} a_i w_i = b \ \ , \ \ w_i \in \{0, 1\} \ \ (i = 1, 2, \ldots, m)$$

We start by considering $\mathbf{LCP}(p, B)$ with

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \ \ , \ \ p = \begin{pmatrix} \phi \\ 0 \\ -1 \\ 1 \end{pmatrix} \quad \text{where } \phi \geq 0$$

Lemma C.1

(i) *$B \in \mathbf{C_0}$.*

(ii) *If $v \in \mathbf{SOL}\,(p, B)$, then, $v_1 \in \{0, 1\}$.*

Proof

(i) Let $x \geq 0$, then $x^T B x = x_1x_2 + x_1x_3 + x_2x_3 \geq 0$.

(ii) Suppose $v \in \mathbf{SOL}\,(p, B)$, then

$$\begin{array}{rcrcr} v_1 \geq 0 & , & v_4 + \phi \geq 0 & , & v_1(v_4 + \phi) = 0 \\ v_2 \geq 0 & , & v_1 \geq 0 & , & v_2 v_1 = 0 \\ v_3 \geq 0 & , & v_1 + v_2 - 1 \geq 0 & , & v_3(v_1 + v_2 - 1) = 0 \\ v_4 \geq 0 & , & -v_1 + 1 \geq 0 & , & v_4(-v_1 + 1) = 0 \end{array}$$

From $-v_1 + 1 \geq 0$ and $v_1 \geq 0$, we get $0 \leq v_1 \leq 1$. However, $v_1 v_2 = 0$ together with $v_1 + v_2 - 1 \geq 0$ implies that if $v_1 > 0$ then $v_1 \geq 1$. Thus, $v_1 \in \{0, 1\}$. ■

Defining $c^i = (a_i, 0, 0, 0)$, $(i = 1, 2, \ldots, m)$ and using p, B as defined above let

$$M = \begin{pmatrix} B & 0 & \cdots & 0 & 0 & (c^1)^T \\ 0 & B & \cdots & 0 & 0 & (c^2)^T \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & B & 0 & (c^m)^T \\ c^1 & c^2 & \ldots & c^m & 0 & 0 \\ -c^1 & -c^2 & \ldots & -c^m & 0 & 0 \end{pmatrix} \quad , \quad q = \begin{pmatrix} p \\ p \\ p \\ \vdots \\ p \\ -b \\ b \end{pmatrix}$$

Proposition C.2

(i) *M is a copositive matrix.*

(ii) *There exists $z = (v^1, v^2, ..., v^m, u_1, u_2)$ (where $v^i \in R^{1\times 4}$) that solves* $\mathbf{LCP}(M, q)$ *if, and only if, $(v_1^1, v_1^2, ..., v_1^m)$ is a solution to the knapsack problem.*

Proof

(i) Let $z = (v^1, v^2, ..., v^m, u_1, u_2) \geq 0$, then $zMz^T = \sum_{i=1}^m v^i B (v^i)^T + u_1 \sum_{i=1}^m c^i (v_i)^T$. So, since $c^i \geq 0$ $(i = 1, 2, \ldots, m)$, and by Lemma C.1, $zBz^T \geq 0$.

(ii) It can be easily verified that if $(w_1, w_2, \ldots, w_m)$ is a solution to the knapsack problem, then $v^i = (w_i, 1 - w_i, 0, 0)$ $(i = 1, 2, \ldots, m)$, $u_1 = 0$, $u_2 = 0$ solves $\mathbf{LCP}(q, M)$.

On the other hand, Suppose that $(v^1, v^2, \ldots, v^m, u_1, u_2)$ solves $\mathbf{LCP}(M, q)$. Then, since $\sum_{i=1}^m c^i (v^i)^T - b \geq 0$ and $\sum_{i=1}^m -c^i (v^i)^T + b \leq 0$ implies that $\sum_{i=1}^m c^i (v^i)^T = b$, we get that $\sum_{i=1}^m a_i v_1^i = b$. Moreover, by Lemma C.1, $v_1^i \in \{0, 1\}$ for $i = 1, 2, \ldots, m$. Thus, $(v_1^1, v_2^2, \ldots, v_m^m)$ solves the Knapsack problem. ■