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DISTRIBUTION OF THE TIME OF THE FIRST k-RECORD

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We compute the first two moments and give a recursive formula for the generating function of the first k-record index for a sequence of independent and identically distributed random variables that take on a finite set of possible values. When the random variables have an infinite support, we bound the distribution of the index of the first k-record and show that its mean is infinite.

1. INTRODUCTION AND SUMMARY

Let $X_1, X_2, \ldots$ be independent and identically distributed finite-valued random variables with probability mass function

$$p_i = P\{X = i\}, \qquad i = 1, \ldots, m.$$

For a fixed positive integer k, $k > 1$, the random variable

$$T = \min\{n : X_n \le X_i \text{ for exactly } k \text{ of the values } i,\ i = 1, \ldots, n\}$$

is called the first *k-record index*. In Section 2 we determine its first two moments and give a recursive formula for its probability generating function, and in Section 3 we present an upper bound on $P\{T > n\}$.

2. EXACT RESULTS

LEMMA 1: *Let $Y_1, Y_2, \ldots$ be independent Bernoulli random variables with*

$$P\{Y_i = 0\} = \lambda = 1 - P\{Y_i = 1\}.$$

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Define

$$N(j) = \min\{n : Y_1 + \cdots + Y_n = j\}$$

and set $M(j) = N(j) - j$. *Then, for* $k \le j$,

$$E[s^{M(j)} \mid Y_k = 1] = \left(\frac{1-\lambda}{1-\lambda s}\right)^{j-1}$$

PROOF: The preceding follows upon noting that, given $Y_k = 1$, $N(j) - 1$ is distributed as a negative binomial random variable with parameters $j - 1$ and $1 - \lambda$. ■

THEOREM 1: *Let* $\lambda_i = p_i / \sum_{j=i}^{m} p_j$, $i = 1, \ldots, m$. *Starting with* $P_m(s) = s^k$, *the probability generating function* $P_1(s) = E[s^T]$ *can be recursively computed from*

$$P_i(s) = \lambda_i s^k + (1 - \lambda_i s) P_{i+1}\left(\frac{s(1-\lambda_i)}{1-\lambda_i s}\right), \qquad i = 1, \ldots, m-1.$$

PROOF: Suppose that the observed random variables have probability mass function

$$P\{X = j\} = \frac{p_j}{p_i + \cdots + p_m}, \qquad j = i, \ldots, m.$$

Let T_i denote the first k-record index, and let $P_i(s) = E[s^{T_i}]$. Also, let I_i be the indicator for the event that the k-record value is not equal to i, and note that this event will occur if and only if the kth observed random variable is not equal to i. Hence, if $I_i = 0$, then $T_i = k$. Also, if $I_i = 1$, then $T_i = T_{i+1} + N_i$, where T_{i+1} is the number of variables greater than i that we need to observe to obtain a k-record and N_i is the number of values equal to i that are observed in that time. Now, it is easy to see that, conditional on $I_i = 1$ and $T_{i+1} = j$, N_i is distributed as the conditional distribution of $M(j)$ given that $Y_k = 1$, where these latter variables are as defined in Lemma 1. Thus, we see that

$$\begin{aligned}
P_i(s) &= \lambda_i s^k + (1-\lambda_i) E[s^{T_{i+1}+N_i}] \\
&= \lambda_i s^k + (1-\lambda_i) E[s^{T_{i+1}} E[s^{N_i} \mid T_{i+1}]] \\
&= \lambda_i s^k + (1-\lambda_i) E\left[s^{T_{i+1}} \left(\frac{1-\lambda_i}{1-\lambda_i s}\right)^{T_{i+1}-1}\right] \\
&= \lambda_i s^k + (1-\lambda_i s) E\left[\left(\frac{s(1-\lambda_i)}{1-\lambda_i s}\right)^{T_{i+1}}\right] \\
&= \lambda_i s^k + (1-\lambda_i s) P_{i+1}\left(\frac{s(1-\lambda_i)}{1-\lambda_i s}\right),
\end{aligned}$$

and the proof is complete. ■

COROLLARY 1: *With* $\alpha_i = 1 - \lambda_i$,

$$E[T] = k + (k-1)\sum_{i=1}^{m-1} \lambda_i$$

and

$$E[T(T-1)] = k(k-1)\left[\lambda_1 + \frac{\lambda_2}{\alpha_1} + \frac{\lambda_3}{\alpha_1\alpha_2} + \cdots + \frac{\lambda_{m-1}}{\alpha_1\cdots\alpha_{m-2}} + \frac{1}{\alpha_1\cdots\alpha_{m-1}}\right].$$

PROOF: Using the notation of Theorem 1, we have shown that

$$P_i(s) = \lambda_i s^k + (1-\lambda_i s)P_{i+1}(h_i(s)),$$

where

$$h_i(s) = \frac{s(1-\lambda_i)}{1-\lambda_i s}.$$

Differentiation yields that

$$P_i'(s) = ks^{k-1}\lambda_i + (1-\lambda_i s)h_i'(s)P_{i+1}'(h_i(s)) - \lambda_i P_{i+1}(h_i(s)).$$

With $\mu_i = E[T_i]$, we obtain, upon evaluating the preceding at $s=1$ and using that $h_i(1) = 1$ and $h_i'(1) = 1/(1-\lambda_i)$, that

$$\mu_i = (k-1)\lambda_i + \mu_{i+1}.$$

Starting with $\mu_m = k$, the preceding gives that

$$\mu_i = k + (k-1)\sum_{j=i}^{m-1}\lambda_j, \qquad i = 1,\ldots,m,$$

and the first part of the theorem is established because $T = T_1$.

Differentiating a second time gives

$$P_i''(1) = k(k-1)\lambda_i + (1-\lambda_i)\left[(h_i'(1))^2 P_{i+1}''(1) + P_{i+1}'(1)h_i''(1)\right] - \lambda_i h_i'(1)P_{i+1}'(1) - \lambda_i P_{i+1}'(1)h_i'(1).$$

Letting

$$\omega_i = P_i''(1) = E[T_i(T_i-1)],$$

we obtain, because $h_i''(1) = 2\lambda_i/(1-\lambda_i)^2$, that

$$\omega_i = k(k-1)\lambda_i + \frac{1}{1-\lambda_i}\omega_{i+1}.$$

Starting with $\omega_m = k(k-1)$, it easily follows that

$$\omega_1 = k(k-1)\left[\lambda_1 + \frac{\lambda_2}{\alpha_1} + \frac{\lambda_3}{\alpha_1\alpha_2} + \cdots + \frac{\lambda_{m-1}}{\alpha_1 \cdots \alpha_{m-2}} + \frac{1}{\alpha_1 \cdots \alpha_{m-1}}\right],$$

where $\alpha_i = 1 - \lambda_i$, and the result is proven. ■

COROLLARY 2: *If* $p_i = P\{X = i\} > 0$ *for all* $i > 0$, *then*

$$E[T] = \infty.$$

PROOF: If X is a random variable with probability mass function p_i, $i > 0$, set

$$\lambda_i = P\{X = i \mid X \geq i\}.$$

Therefore,

$$\prod_{i=1}^{n} (1 - \lambda_i) = P\{X > n\},$$

implying that

$$\prod_{i=1}^{\infty} (1 - \lambda_i) = 0.$$

However, this implies that

$$\sum_{i=1}^{\infty} \lambda_i = \infty.$$

Now, a simple coupling argument shows that if we have a sequence of independent observations that take on the values $1, \ldots, m-1, m$ with probabilities $p_1, \ldots, p_{m-1}, \sum_{j=m}^{\infty} p_j$ then the expected time until a k-record occurs is shorter for this sequence than it is for the sequence X_j, $j \geq 1$. Hence, from Theorem 1, we obtain that, for all m,

$$E[T] \geq k + (k-1) \sum_{i=1}^{m-1} \lambda_i,$$

and the result follows by letting $m \to \infty$. ■

Remarks:

1. It follows from Ignatov's theorem (see Engelen, Thommassen, and Vervaat [1], Ross [2], and Samuels [3]), which states that the processes of k-record values are independent and identically distributed processes for $k \geq 1$, that $P\{T < \infty\} = 1$.
2. When the random variables are continuous, it is easy to check that

$$P\{T = n\} = \frac{k-1}{n(n-1)}, \quad n \geq k,$$

and so $E[T] = \infty$.

3. BOUNDS ON THE DISTRIBUTION OF T

Suppose that the X_j are positive integer-valued random variables with mass function $p_i = P\{X = i\}$, $i > 0$, and again let T denote the index of the first k-record.

THEOREM 2:

$$P\{T > n\} \le \sum_{i=1}^{\infty} p_i \sum_{r=0}^{k-1} \binom{n}{r} (p_i/\lambda_i)^r (1 - p_i/\lambda_i)^{n-r}.$$

PROOF: Let S_j denote the index of the kth observed value that is at least as large as j, and let $V_j = X_{S_j}$ be the value. Also, let $V = X_T$ be the k-record value. Then, because Ignatov's theorem yields that $P\{V = i\} = p_i$, we have the following:

$$\begin{aligned} P\{T > n\} &= \sum_i P\{T > n \mid V = i\} p_i \\ &= \sum_i p_i P\{S_i > n \mid V_i = i,\ V_j \neq j,\ j = 1, \ldots, i-1\}, \end{aligned}$$

where the final equality uses the fact that the k-record value will equal i if the kth value to be at least i is equal to i and the same cannot be said of any of the values j, $j < i$. Now, knowing that $V_j \neq j$, $j = 1, \ldots, i-1$, makes it more likely that the data values are at least as large as i, which makes the conditional distribution of S_i stochastically smaller than it would be otherwise. Hence, from the preceding we see that

$$\begin{aligned} P\{T > n\} &\le \sum_i p_i P\{S_i > n \mid V_i = i\} \\ &= \sum_i p_i P\{S_i > n\} \\ &= \sum_i p_i \sum_{r=0}^{k-1} \binom{n}{r} (p_i/\lambda_i)^r (1 - p_i/\lambda_i)^{n-r}, \end{aligned}$$

which completes the proof. ∎

The following corollary is immediate.

COROLLARY 3: *Let B_i, $i \ge 1$, be negative binomial random variables with respective parameters $(k, p_i/\lambda_i)$, and let X be independent of the B_i and such that $P\{X = i\} = p_i$, $i \ge 0$. Then,*

$$T \le_{st} B_X.$$

When $p_i = 0$, for $i > m$, the preceding corollary can be used to give bounds on the moments of T. For instance, it yields that

$$E[T] \le k \sum_{i=1}^{m} \lambda_i,$$

which can be compared to the exact result given in Theorem 1, namely,

$$E[T] = k\sum_{i=1}^{m}\lambda_i - \sum_{i=1}^{m-1}\lambda_i.$$

For the second moment, the bound is

$$E[T^2] \le \sum_{i=1}^{m} p_i\left[(k\lambda_i/p_i)^2 + k(1 - p_i/\lambda_i)(\lambda_i/p_i)^2\right]$$

$$= k(k+1)\sum_{i=1}^{m}\lambda_i^2/p_i - k\sum_{i=1}^{m}\lambda_i,$$

which can be compared to the exact expression given by Theorem 1.

References

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