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ADVANTAGEOUS PROPERTIES OF DUAL TRANSHIPMENT POLYHEDRA*

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Abstract. The dual linear programs for the transhipment problem over a directed graph, $G = \{N, E\}$, are shown to have polyhedra with properties that make them well suited to vertex visiting solution techniques, like the simplex method. In particular, nondegenerate cases are shown to have feasible regions with considerably fewer extreme points than the feasible sets for primal problems. The adjacency structure of feasible bases is also shown to be quite favorable. In fact, the Hirsch Conjecture is valid when the network is complete. A dual-based simplex method for cases of the transhipment problem, which include the shortest paths problem, is presented that finds an optimal solution in no more than Min $(|E| - |N| + 1, |N||N - 1|/2)$ pivots.

Key words. linear programming, network flows, hirsch conjecture

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1. Introduction. Consider a directed graph, G, having a node set,

$$N = \{0, 1, \cdots, m\}$$

and an edge set, $E = \{e_k = (i, j) \mid \textit{edge } k \textit{ is directed from } i \textit{ to } j,\ k = 1, \cdots, n\}$. Let the vector $b \in \mathbb{R}^m$ represent a demand for flow at nodes $1, \cdots, m$. When $b_i < 0$, we say that there is a supply at node i. Let c_{ij} represent the cost of sending one unit of flow across edge (i, j). The transhipment problem is defined as the task of finding an assignment of flow to the edges that satisfies the demand at each node at minimum total cost.

A commonly used linear programming (LP) formulation for the transhipment problem is:

$$\text{TP: Min} \sum_{(i,j)\in E} c_{ij}x_{ij}$$

$$\text{subject to: } \sum_{i:(i,j)\in E} x_{ij} - \sum_{k:(j,k)\in E} x_{jk} = b_j \qquad j = 1, \cdots, m$$

$$x_{ij} \geqq 0.$$

Its dual LP, (DTP), is formulated as follows:

$$\text{DTP: Max} \sum_{i=1}^{m} b_i y_i$$

$$\text{subject to: } y_j - y_i \leqq c_{ij} \qquad (i,j) \in E$$

$$y_0 = 0.$$

A constraint for node 0 is not included in the primal formulation since it would be redundant. For consistency, we set the (dummy) dual variable y_0 equal to 0. Let $A \in$

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$\mathbb{R}^{m\times n}$ be the node-arc incidence matrix for G, without node 0. In particular,

$$a_{ik}=\begin{cases} -1 \text{ if } e_k \text{ is directed out of node } i \\ +1 \text{ if } e_k \text{ is directed into node } i \\ 0 \text{ otherwise.} \end{cases} \qquad i=1,\cdots,m, \quad k=1,\cdots,n$$

The feasible region for TP can be represented by the system, $Ax=b$, $x\geqq 0$, where A has row rank m and in each column there is no more than one $+1$ and no more than one -1.

In [Tard85], an algorithm is presented which solves the transhipment problem in $O(n^2m^3\log m)$ arithmetic operations, thus resolving the long standing question of whether there exists an algorithm solving the problem in a number of steps that is bounded by a polynomial in n and m and is independent of the values of b_i and c_j, (i.e., *a strongly polynomial algorithm*). The method can be described as a primal-based algorithm since the iterates generated by it satisfy the constraints of TP. A dual framework to the algorithm has been developed in [Fuji85] and improved upon in [GaTa86] giving an $O(m^4\log m)$ algorithm that solves the problem by way of the dual. This represents a significant theoretical improvement over the primal algorithm since n is of order m^2 in dense networks.

A Simplex method for the Transhipment problem is described in [Cunn76]. Here, the method for avoiding cycling under degeneracy by visiting only *strongly feasible bases* is introduced. (See [Cunn79] for a description of its theoretical underpinnings.) In [Orl85], the author shows that, in contrast to more general cases, the simplex method using Dantzig's pivot rule solves instances of TP having upper bounds on the variables in no more than $O[(u)mn^2\log n)]$ pivots, when solving the problem by way of the primal formulation, (where u represents the largest upper bound on a variable). For the dual formulation of the problem, Orlin ([Orl84]) presents a Simplex method that requires no more than $O(m^3\log n)$ pivots, giving a strongly polynomial vertex visiting algorithm.

In an attempt to explain why the dual formulation appears to be better suited to vertex visiting methods, like the simplex method, considerable attention has been paid to linear programs for the dual of the transportation and assignment problems (DTAP). These problems arise from a transhipment problem over a bipartite graph with node sets N_1 and N_2 and edge set $\{(i,j)\mid i\in N_1, j\in N_2\}$. (In the assignment problem, $|N_1|=|N_2|$.) It had been shown that the polyhedra associated with DTAPs have considerably fewer vertices than those of the associated primal problems ([BaRu84]). The polyhedra have also been shown to satisfy the Hirsch Conjecture ([Bal84]). Coupled with these findings was the development of dual based methods for the problems having rather good performance guarantees. Goldfarb ([Gold85]) and Balinski ([Bal86]) give simplex algorithms solving the assignment problem that require no more than $O(|N_1|^3)$ operations. Similar techniques have been used to solve special cases of the Transportation problem in $O(|N_1|^2|N_2|+|N_1||N_2|^2)$ time ([KLS87]).

In the following sections, we analyze more general cases of the transhipment problem and show that the dual formulation has advantages over the primal for these problems as well. In § 2, we list some special properties of TP and DTP. We present a discussion about what happens when the node-arc incidence matrix A is a *Leontief Matrix*. An interesting case is the one in which A represents a complete directed graph, where between every pair of nodes i and j there exists a symmetric pair of edges (i,j) and (j,i). In § 3, we show that the dual LPs for this case have considerably fewer basic feasible solutions than those of the primal, regardless of the values of b and c. In § 4, we comment on the adjacency structure of the feasible bases of DTP. We show that the *distance* between any two extreme points is relatively small. In fact, the Hirsch Conjecture is shown to be valid for instances of DTP occurring over a complete directed graph. In § 5, we exploit these results to give a simplex method for cases of DTP which include the dual to the shortest paths problem, which require no more than

$$\mathrm{Min}\,(|E| - |N| + 1,\ |N|\,|N-1|/2)$$

pivots.

2. Preliminaries. Let A_{*j} denote the jth column of A. For a set $S \subset \{1 \cdots n\}$, A_{*S} is defined as the submatrix of A consisting only of those columns with an index in S. A *basis* is defined as a set of m linearly independent columns of A. If A_{*S} is a basis, it is said to be *feasible* for TP if the unique (basic) solution to the system, $A_{*S}x_S = b$, is such that $x_S \geqq 0$. This occurs if $A_{*S}^{-1}b$ is nonnegative. The basis A_{*S} is said to be feasible for DTP if the unique solution, y_S, to $y^T A_{*S} = c_S$, is such that

$$y_S^T A_{*j} \leqq c_j, \quad \text{for all } j \notin S.$$

We say that the basis is *nondegenerate* if this set of inequalities holds strictly. If the basis is nondegenerate, the solution corresponds to a nondegenerate vertex of the polyhedron represented by the constraints of the problem. In such a case, no other basis could determine the solution y_S.

For the remainder of our discussion, we will assume that the vector c is such that there exists at least one dual feasible basis, i.e., that the network is connected and there are no directed cycles of negative total cost (so the optimal objective function value for TP is finite).

It is well known that a set of columns in A are linearly independent if the set of edges in the network associated with them do not form any cycles. So every collection of edges forming a spanning tree of the network corresponds to a basis. We will say a spanning tree is *primal feasible* or *dual feasible*, if it corresponds to a feasible basis for TP or DTP, respectively.

If T is a spanning tree, the subtree of T rooted at node i, T_i, has node set

$$N_{T_i} = \{\, j \in N \mid \textit{the unique path in } T \textit{ from } 0 \textit{ to } j \textit{ contains node } i \,\}.$$

A directed edge (i, j) in T is *downward* if the path from 0 to j in T contains node i and *upward* if it does not. A tree consisting solely of downward edges is called a *branching* rooted out of node 0. For a partition of the nodes into sets C and $\bar{C}$, where node 0 is in C, we define the edges that are directed from a node in C to a node in $\bar{C}$ to be the *forward edges* of the cut, $\{C, \bar{C}\}$, and the edges directed from $\bar{C}$ to C to be the *backward edges* of the cut $\{C, \bar{C}\}$.

When the matrix A has no more than one positive element in any column it is called a *Pre-Leontief Matrix*. The system $Ax = b$, $x \geqq 0$ is called a *Pre-Leontief Substitution System* if, in addition, b is positive. When there is at least one solution to the system, we say that the system is a *Leontief Substitution System* (LSS) and A is a *Leontief Matrix*.

The properties of LSS's and their bases are discussed in great detail in [Dant55] and [Vein68]. LSS's have many characteristics not found in general LP's, which makes them worthy of special consideration.

We say that column j is in *substitution class* i, when the positive element of A_{*j} is in the ith row, and in *substitution class* 0, if $A_{*j} \leqq 0$. The feasible bases for an LSS are such that there is exactly one member from each of Substitution Classes $1, \cdots, m$. In addition, every feasible basis of an LSS has a nonnegative inverse, so if there is a solution for some $b > 0$, then there is a solution for any $b \geqq 0$.

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We find the following property particularly useful.

PROPOSITION 1. *If A is a Leontief Matrix, then $y^TA \leqq c$, has a finite "maximal solution," y^*, for all c for which the system is feasible. That is, every solution to the system, $\hat{y}$, is such that $\hat{y} \leqq y^*$. Furthermore, the system $y^TA \geqq c$, has a finite "minimal solution" for all c for which it is feasible.*

A proof of this result can be found in [CoVe72], where polyhedral sets with minimal (maximal) solutions are characterized.

If TP is feasible when $b > 0$, then A is a Leontief Matrix with a column of belonging to substitution class i if the associated edge is directed into node i. So the feasible bases for TP with $b > 0$ correspond to spanning trees with each node (except node 0) having in-degree one, which is a branching rooted out of node 0. If TP is feasible when $b < 0$, then the system, $(-A)x = -b$, $x \geqq 0$ is an LSS, with substitution classes corresponding to the out-edges of nodes $1, \cdots, m$, so feasible bases correspond to branchings rooted into node 0. Consequently, if the network contains at least one branching rooted out of (into) node 0, then, by Proposition 1, the system in DTP contains a maximal (minimal) solution.

3. The number of feasible bases of TP, DTP.

In common worst-case analyses of vertex visiting algorithms, a relatively small number of feasible bases for a class of LP problems would indicate that the problems in the class are likely to be solved more quickly by the simplex method than more general problems.

Since the bases for transhipment problems must correspond to spanning trees in the network, we are able to provide the following bounds on the number of feasible bases for TP and its dual.

LEMMA 1. *The number of feasible bases for any nondegenerate instance of* TP *arising from a directed complete graph with $|N|$ nodes is exactly $|N|^{|N|-2}$.*

Proof. From Cayley's Theorem, we know that an undirected complete graph with $|N|$ nodes contains $|N|^{|N|-2}$ spanning trees. Given one such tree, we find an associated feasible basis for TP as follows. If node i and node j are linked in the tree, and i is in the path from node 0 to node j, we choose for the basis the column of A corresponding to edge (i, j) if $\sum_{k \in N_{T_j}} b_k \geqq 0$ or edge (j, i) if $\sum_{k \in N_{T_j}} b_k \leqq 0$.

When $b > 0$, for instance, all edges in the tree are directed downward, which is a branching rooted out of node 0. If TP has degenerate bases, the sum of the demands at a subtree of some tree is equal to 0. We could associate more than one feasible basis with such a tree. □

The number of spanning trees in the network which are dual feasible, it turns out, is considerably smaller than that of the primal, in fact, the following holds.

LEMMA 2. *The number of feasible bases for a nondegenerate, feasible instance of* DTP *arising from a directed complete graph with $|N|$ nodes can be as small as $|N|$.*

Proof. Since the graph is complete, it is guaranteed to contain a branching rooted into node 0 and a branching rooted out of node 0. So there are feasible bases corresponding to a minimal solution, y, and a maximal solution, $\bar{y}$. The polytope for DTP is contained within the region, $\{z \mid y \leqq z \leqq \bar{y}\}$. Any bounded, nondegenerate, m-dimensional polyhedron, such as this one, must contain at least $m + 1 = |N|$ affinely independent extreme points.

The tightness of the bound follows from the case of DTP where

$$c_{ij} = \begin{cases} 1 & j = i+1 \text{ or } (i,j) = (m,0) \\ M & \text{otherwise} \end{cases} \qquad i,j = 0, \cdots, m$$

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where M is a number larger than m. Notice that, by summing the constraints associated with the set of edges, $\{(i, i+1), (i+1, i+2), \cdots, (j-1, j)\}$, and the set, $\{(j, j+1), \cdots, (m, 0), \cdots, (i-1, i)\}$, we obtain implicit constraints

$$-m+(j-i-1) \leqq (y_j - y_i) \leqq j-i$$

for every pair of nodes, i and j, where $i<j$. This implies that all but the $m+1$ constraints associated with edges, $(0, 1), (1, 2), \cdots, (m, 0)$ are redundant. The $m+1$ basic feasible solutions for this case of DTP are enumerated as follows:

$$y_i^K = \begin{cases} i & i<K \\ -m+i-K & i \geqq K \end{cases} \quad i=1, \cdots, m \quad K=1, \cdots, m+1. \qquad \square$$

The number of feasible bases for DTP is different for differing values of the vector c. This is in contrast to the dual of transportation and assignment problems, (DTAPs). In [BaRu84], the authors show that the feasible bases of any nondegenerate instance over a complete bipartite graph, $B = \{N_1, N_2, (N_1 \times N_2)\}$, are in one-to-one correspondence with a set of *signature vectors*, $s \in \mathbb{Z}^{|N_1|}$, satisfying $s_i > 0$, and $\sum_{i=1}^{|N_1|} s_i = |N_1| + |N_2| - 1$. So the number of feasible bases is $\binom{|N_1|+|N_2|-2}{|N_1|-1}$.

THEOREM 1. *The tightest upper bound on the number of feasible bases for a nondegenerate, feasible instance of* DTP *arising from a directed complete graph with* $|N|$ *nodes is* $\binom{2(|N|-1)}{|N|-1}$.

Proof. Consider the following dual formulation of an instance of the assignment problem, where $N_1 = N_2 = \{0, \cdots, m\}$:

$$\hat{D}: \text{Max } e^T u - e^T v$$

$$\text{subject to: } u_j - v_i \leqq c_{ij} \qquad i \in N_1, \quad j \in N_2, \quad i \neq j$$

$$u_j - v_i \leqq 0 \qquad i \in N_1, \quad j \in N_2, \quad i = j$$

$$v_0 = 0.$$

Independent of the values in c, the polytope associated with $\hat{D}$ has no more than $\binom{2m}{m} = \binom{2(|N|-1)}{|N|-1}$ vertices. One of the faces of the polytope,

$$F = \{(u, v) \mid u_j - v_j = 0, j = 0, \cdots, m\},$$

is equivalent to the nondegenerate, feasibility set of DTP, thus the number of feasible bases for DTP is bound as stated.

The upper bound is tight when $c_{ij} = i(|N| - j)$, since a one-to-one correspondence can be drawn between the feasible bases for this case and a set of *degree vectors*, $d \in \mathbb{Z}^{m+1}$, satisfying $d_i \geqq 0$, and $\sum_{i=0}^{m} d_i = |N| - 1$. There are $\binom{2(|N|-1)}{|N|-1}$ such vectors.

Let $d = (d_0, \cdots, d_m)$ be some such degree vector. Notice that at least one element from this vector must be equal to 0. So let $d = \{d_S, d_T\}$, where $S = \{k \mid d_k > 0\}$ and $T = \{k \mid d_k = 0\}$.

Consider the following subset of the constraints from the instance of DTP:

$$y_j - y_i \leqq i(|N| - j) \qquad i \in S, \quad j \in T$$

$$y_0 = 0.$$

Notice that this set of constraints resembles that of the dual formulation of some instance of the Transportation problem (DTAP), which is known to be nondegenerate since it is equivalent to a *cross-free* case, as described in [BaRu84]. Notice also that d_S constitutes

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a valid signature vector for this DTAP. Let $z = \{z_S, z_T\}$ be the unique basic solution associated with this signature.

If z also satisfies the remaining inequalities from the instance of DTP, we can conclude that it is a basic feasible solution to the instance at hand. If the constraints are satisfied with strictness, then z is a unique, nondegenerate solution corresponding with d.

The constraints that are binding at z correspond to edges that comprise a tree spanning nodes $0, \cdots, m$. Since a tree is a connected subgraph, we know that the following is true.

For any $j \in T$ there exists an $\hat{i} \in S$ such that

$$z_j - z_{\hat{i}} = \hat{i}(|N| - j),$$

$$z_k - z_{\hat{i}} \leqq \hat{i}(|N| - k) \quad \textit{for all other } k \in T$$

implying that, $z_k - z_j \leqq \hat{i}(j - k)$. Since $0 \leqq \hat{i}, j, k < |N|$, we can conclude that $z_k - z_j < j(|N| - k)$. $j, k \in T$.

Also, for any $i \in S$ there exists a $\hat{j} \in T$ such that

$$z_{\hat{j}} - z_i = i(|N| - \hat{j}),$$

$$z_{\hat{j}} - z_h \leqq h(|N| - \hat{j}) \quad \textit{for all other } h \in S$$

so, $z_i - z_h \leqq (h - i)(|N| - \hat{j})$. Since $0 \leqq \hat{j}, i, h < |N|$, we see that

$$z_i - z_h < h(|N| - i). \qquad i, h \in S.$$

And, for any pair $h \in S$, $k \in T$ there exist $\hat{i} \in S$ and $\hat{j} \in T$ such that

$$z_k - z_{\hat{i}} = \hat{i}(|N| - k), \qquad z_{\hat{j}} - z_h = h(|N| - \hat{j})$$

where

$$z_{\hat{j}} - z_{\hat{i}} \leqq \hat{i}(|N| - \hat{j})$$

so, $z_h - z_k \leqq \hat{i}(k - \hat{j}) - h(|N| - \hat{j}) < k(|N| - h)$.

Therefore, the all of the constraints of this instance of DTP are satisfied; and z is one of $\binom{2(|N|-1)}{|N|-1}$ nondegenerate, basic feasible solutions. □

COROLLARY. *The number of d-dimensional faces in the polytope for* DTP *arising from a directed complete graph with* $|N|$ *nodes, when feasible and nondegenerate, is no more than*

$$\binom{2(|N|-1)-d}{|N|-1}\binom{|N|-1}{d}.$$

Proof. The number of bounded faces of the polytope for an instance of DTAP has been established in [BaRu84]. This bound holds for DTP since its entire feasible region is on face F of the DTAP, $\hat{D}$, described above. When $c_{ij} = i(m + 1 - j)$, all of the vertices, hence all of the bounded faces of the polytope of $\hat{D}$ lie on F, so the bound is tight. □

We wish to point out that the face, F, described in the proof, is the optimal face for problem $\hat{D}$, whenever c is such that DTP is feasible. (Notice that the value of the objective function must be less than zero whenever $u_j < v_j$ for some j.) Therefore, a (Phase I) feasible starting basis for DTP can be found by way of an efficient strongly polynomial simplex method by solving $\hat{D}$ using either the method of [Bal86] or [Gold85].

4. The "distance" between feasible bases of DTP. Two feasible bases, B and $\hat{B}$, are *adjacent* if they have $m - 1$ columns of A in common. To travel from feasible basis

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B to an adjacent basis requires a *pivot*. To describe a pivot for DTP, consider the formulation with the addition of slack variables:

$$\text{DTP: Max} \sum_{i=1}^{m} b_i y_i$$

$$\text{subject to } y_j - y_i + z_{ij} = c_{ij} \qquad (i,j) \in E$$

$$y_0 = 0 \qquad z_{ij} \geqq 0.$$

If (i, j) is in the tree T associated with feasible basis B then $z_{ij} = 0$. Increasing some z_{ij} from 0 corresponds to dropping (i, j) from T, which cuts the nodes into sets, C and $\bar{C}$ (where node 0 belongs to C).

If (i, j) is an upward edge, then $\bar{C}$ is the set of nodes in the subtree of T rooted at node i. To maintain feasibility, any increase in z_{ij} of Δ must be accompanied by an increase of Δ in each of y_k, $k \in \bar{C}$. This results in a $\Delta(\sum_{k \in \bar{C}} b_k)$ change in the objective function value.

Thus, the (possibly negative) *reduced cost* of dropping (i, j) from the tree is equal to the total demand for flow at the subtree rooted at node i.

As y_k, $k \in \bar{C}$ increases, the slack in the constraints associated with the forward edges of $\{C, \bar{C}\}$ decreases. If no forward edge exists, then Δ can be made arbitrarily large, without violating any constraints. Otherwise, Δ is limited by the slack in the constraints associated with these edges. When a constraint becomes binding, we obtain a new solution, $\hat{y}$. The now binding constraint is called the *entering* constraint, giving a new basis $\hat{B}$ and dual feasible tree $\hat{T}$.

If there is degeneracy, there may be more than one candidate for the entering constraint during a pivot. It is also possible that some candidate entering constraint is already binding, so a pivot may occur without changes in the values of y and z.

If (i, j) is a downward edge, a pivot can be similarly described. The reduced cost of dropping (i, j) from the tree is equal to the total supply at the nodes in the subtree rooted at node j. In a nondegenerate pivot, the entering edge is backward in the cut $\{C, \bar{C}\}$. Under degeneracy, however, there may exist candidate entering edges which are forward.

The *distance* between two feasible bases is defined to be the smallest number of pivots necessary to get from one to the other. If the distance between every pair of bases for an LP is relatively small, then there is reason to believe that the LP is fairly well suited to some vertex visiting method.

Suppose some instance of TP is feasible for $b > 0$. Then, as we have seen, there is a dual feasible basis B^*, which determines a maximal solution to DTP, y^*, and has an associated tree T^*, which is a branching rooted out of node 0.

For some dual feasible basis, B^k, with associated tree, T^k, and solution, y^k, let

$$S^k = \{0\} \cup \{i \in \{1, \cdots, m\} \mid \textit{there exists a path from } 0 \textit{ to } i \textit{ in} T^k \cap T^*\}.$$

Note that $i \in S^k$ implies that $y_i^k = y_i^*$.

LEMMA 3. *If* $(i, j) \in T^k$ *and* $i \in S^k$ *then* $y_j^k = y_j^*$.

Proof. Since $(i, j) \in T^k$ and $i \in S^k$, $y_j^k - y_i^k = c_{ij} = y_j^k - y_i^*$. Since y^* is a feasible solution, $y_j^* - y_i^* \leqq c_{ij}$. So, $y_j^k \geqq y_j^*$. But since y_j^* is maximal, $y_j^k = y_j^*$. □

LEMMA 4. *If* $(i, j) \in T^*$ *and* $y_j^k = y_j^*$ *then* $y_i^k = y_i^*$.

Proof. Since $(i, j) \in T^*$ and $y_j^k = y_j^*$, $y_j^* - y_i^* = c_{ij} = y_j^k - y_i^*$. Since y^k is a feasible solution, $y_j^k - y_i^k \leqq c_{ij}$. So, $y_i^k \geqq y_i^*$. But since y_i^* is maximal, $y_i^k = y_i^*$. □

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THEOREM 2. *Suppose B^* is a feasible basis for* DTP *arising from a network with $|N|$ nodes and $|E|$ edges. If B^* corresponds to a branching rooted out of node* 0, *then the distance between B^* and any other feasible basis, say B^0, is bounded by*

$$\min\left(|E| - |N| + 1, \binom{|N|}{2}\right).$$

Proof. We construct a sequence of pivots that starts with B^0 and terminates with B^* which is no longer than the bound stated.

Suppose k pivots have been performed, giving dual feasible solution y^k and dual feasible tree, T^k. While $T^k \neq T^*$, choose as the *next edge* to drop from T^k any edge (i, j) such that exactly one of i or j is a member of S^k.

An edge (p, q) is chosen from the set of candidate entering edges, which is forward in the cut $\{C, \bar{C}\}$, giving a new tree, T^{k+1}, and a new feasible solution, y^{k+1}. (We know that forward (p, q) exists, else the branching T^* doesn't exist.)

There are three possible occurrences, shown in the following cases.

Case 1. $p \notin S^k$. Then, $S^{k+1} = S^k$. While this event does not contribute to the progress toward B^*, we will make assurances that (i, j) is not an entering edge in any subsequent pivots. (Notice that (i, j) could never enter again if it is an upward edge, since the only entering edges are either directed away from S^{k+1} or are disjoint from it.)

Case 2. $p \in S^k$ and $(p, q) \in T^*$. Then with respect to T^{k+1}, $q \in S^{k+1}$, so $|S^{k+1}| \geqq |S^k| + 1$.

Case 3. $p \in S^k$ and $(p, q) \notin T^*$. In this case, (p, q) may have been an edge in a previous tree, so we would rather not allow this edge to enter. We choose to select a different entering edge that satisfies either Case 1 or Case 2. We will show, with respect to the new solution y^{k+1}, that such an edge exists.

From Lemma 3 we know that since (p, q) is a candidate entering edge, y_q^{k+1} must be equal to its terminal value y_q^*, which implies that the solution y^{k+1} is degenerate, (i.e., there is more than one basis corresponding to y^{k+1} and (p, q) is not the only allowable downward entering edge for the pivot). An alternative entering edge can be found as follows:

> Set $h = q$.
> Let (g, h) be the unique edge in T^* directed into node h. By Lemma 4, $y_g^{k+1} = y_g^*$. If $g \notin \bar{C}$, then (g, h) is an acceptable entering edge. Otherwise, set h equal to g and repeat.

Suppose, initially, that T^0 has no edges in common with T^*. By construction, edges belonging to T^* are never to be dropped. Since there are no more than $|E| - |N| + 1$ edges not belonging to T^*, and no edge can be dropped more than once, the distance between B^0 and B^* is bounded by $|E| - |N| + 1$.

For the case when the graph is dense, we should notice that if $|S^0| = s$, then the number of valid next edges is bounded by $t = |N| - s$. It is impossible for all t of the valid next edges to be dropped, without an occurrence of Case 2, (otherwise there would be no edges in T^* joining S^t to $\bar{S}^t$, which is impossible since T^* is a spanning tree).

So, if $|S^0| = 1$, (i.e., $S^0 = \{0\}$), there can be no more than $|N| - 1$ consecutive occurrences of Case 1. Hence, the total number of pivots, K, guaranteeing that $|S^K| = |N|$ is bounded by $|N| - 1 + |N| - 2 + \cdots + 1 = \binom{|N|}{2}$. $\square$

We point out that if $\hat{B}$ is a dual feasible branching rooted into node 0, it corresponds to the minimal solution of the system, $y^T A \leqq c$. Since the feasible bases of the system, $y^T A \leqq c$ are in one-to-one correspondence with the feasible bases of the system,

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$y^T(-A) \leqq c$, $\hat{B}$ could be found by a similar sequence of pivots as is used to find the basis for the maximal solution of $y^T(-A) \leqq c$.

Also, for many transhipment problems, the choice of root node 0 is arbitrary. By renumbering the nodes, we see that the basis for a dual feasible branching rooted out of or into some node r, if one exists, is never more than $\min(|E| - |N| + 1, \binom{|N|}{2})$ pivots away from the other dual feasible bases.

COROLLARY. *If some feasible basis of* DTP, *say* B^0, *has k columns of A in common with the basis for a dual feasible branching, then the distance between the two bases is bounded by* $\min(|E| - |N| + 1, \binom{|N|-k}{2})$.

Proof. Even if $S^0 = \{0\}$, there cannot be more than $|N| - k - 1$ valid next edges, since edges that are in T^* are never dropped. So the total number of pivots necessary to guarantee that $|S^K| = |N|$, is bounded by

$$|N| - k - 1 + |N| - k - 2 + \cdots + 1 = \binom{|N| - k}{2}. \qquad \square$$

Nondegenerate polytopes of dimension d, being determined by k linear inequalities, are said to satisfy the Hirsch Conjecture if the distance between any pair of feasible bases is bounded by $k - d$. Cases of TP and DTP known to have polytopes satisfying the Hirsch Conjecture include the assignment problem ([Bal74]), special cases of the transportation problem ([BaRu74]), DTAPs over complete bipartite graphs ([Bal84]), the shortest path problem ([Saig69]), and more generally, TPs that are also LSSs ([Grin71]). The above results allow us to extend this list to include some other cases of DTP.

THEOREM 3. *The polytope for* DTP *arising from a directed complete graph with* $|N|$ *nodes satisfies the Hirsch conjecture. That is, the distance between any two feasible bases is bounded by* $(|N| - 1)^2$.

Proof. Let B^1 and B^2 be dual feasible bases. Since the graph is complete, TP is feasible when $b > 0$ and when $b < 0$, so DTP has feasible bases corresponding to branchings rooted out of and into node 0. By Theorem 2, the distance between B^1 and any of these bases is bounded by $\binom{|N|}{2}$.

Let e be an edge incident to node 0 in the tree, T^2, associated with B^2. If e is directed into (out of) node 0 then, by Lemma 3, T^2 shares edge e with some dual feasible branching, $\hat{B}$, rooted into (out of) node 0. By the Corollary of Theorem 2, the distance between B^2 and $\hat{B}$ is bounded by $\binom{|N|-1}{2}$. So the distance between B^1 and B^2 is bounded by $\binom{|N|}{2} + \binom{|N|-1}{2} = (|N| - 1)^2$. $\square$

5. The Simplex method for special cases of DTP. Having established that the feasible bases for the instance of DTP with n edges and $m + 1$ nodes are no more than m^2 pivots away from each other, we now attempt to determine just how well a simplex method is expected to perform on particular instances of the problem. Ideally, there would be a simple rule for choosing the entering and leaving variables for the pivots, which would follow the sequence of bases of minimum length, while improving the objective function at each step. For general LPs, there is no guarantee that such rules exist. (For DTP however, there is a rule which follows a sequence to an optimal basis containing no more than $O(m^3 \log m)$ pivots [Orl84].)

Let us divert our attention to a special case of the transhipment problem called the shortest (0 to all i) paths problem (SP), which is formulated as follows:

$$\begin{array}{ll} \text{SP: Min } c^T x & \text{DSP: Max } e^T y \\ \text{subject to } Ax = 1 & \text{subject to } A^T y \leqq c \\ \qquad x \geqq 0 & \end{array}$$

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This problem has been shown to be easily solved by variants of Dijkstra's method ([Dijk59]). Until the time of this writing, the best simplex method requires $O(m^2 n \log m)$ pivots for the primal formulation and $O(m^3 \log m)$ pivots for the dual formulation, ([Orl85]). For this and other cases of TP and DTP we are able to prove the following theorem.

THEOREM 4. *When* $b > 0$, *as in, for example, the shortest paths problem, an optimal solution to* DTP (DSP) *can be found by the simplex method, starting from any feasible basis, after no more than* $\min(|E| - |N| + 1, \binom{|N|}{2})$ *pivots, by using Dantzig's Pivoting Rule.*

Proof. When $b > 0$, feasible bases corresponding to the maximal solution, y^*, are optimal. Starting from feasible basis, B, with corresponding tree, T, the result holds if Dantzig's rule chooses to drop an edge that is a valid next edge as is defined in the proof of Theorem 2. This would have to be the edge with the largest reduced cost. Recall that it will be acceptable to terminate with any dual feasible basis that corresponds to a branching rooted out of node 0.

As we have seen, the reduced cost of dropping an upward tree edge, (i, j), is equal to the total flow demanded from the subtree rooted at i, which is positive when $b > 0$. (When all $b_k = 1$, the reduced cost is precisely the number of nodes in the subtree at node i.)

Let $S = \{0\} \cup \{i \in \{1, \cdots, m\}|$. *The path from* 0 *to* i *in* T *consists of downward edges.*$\}$

Suppose that the edge $(i, j) \in T$ has the largest reduced cost. Edge (i, j) must be an upward edge, where node j belongs to S, otherwise the path in T from node 0 to j would have to contain some other upward edge, (g, h). In that case, both of nodes i and j would be in the subtree of T, rooted at g, contradicting that (i, j) has the largest reduced cost.

Since all of the dropped edges are upward, we can always choose an entering edge that is forward if a branching exists. This edge satisfies either Case 1 or Case 2 as in the proof of Theorem 2, so (i, j) is a valid next edge. The Simplex method terminates when there are no more upward tree edges, so the terminal basis corresponds to a dual feasible branching, which is optimal. $\square$

When $b \geqq 0$, feasible bases corresponding to the maximal solution, y^*, are also optimal. In this case, however, it is possible that some edge with the largest reduced cost is not a valid next edge. By modifying the rule so that the edge with the largest reduced cost, which also has one of its nodes belonging to the set S, is chosen, we could solve this case of TP with the same performance bound as above.

Notice that cases of DTP where $b \leqq 0$ can be solved analogously. These cases can be posed as minimization problems where the next edge chosen is the one with the smallest reduced cost.

6. Remarks. The number of basic feasible solutions and the distance between them has been used as a yardstick on how well or how poorly we could expect a vertex visiting method to perform on a given problem. Since, when using a simplex method, each pivot must result in an improvement to the value of the objective function, (improvements of 0 are acceptable), other interesting measures for the suitability of an LP to the method can be introduced. They include the longest and shortest *monotonic distance* between feasible bases. This is, respectively, the length of the longest and shortest sequence of pivots between two feasible bases, when we insist on the monotonicity of some linear function. The simplex method could do no better than travel to the optimal basis along the shortest monotonic path and no worse than travel along the longest path, so these measures would be quite useful in the attempt to provide tighter bounds on the expected performance of a simplex method for these problems.

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Similar measures have yet to be developed for judging the relative suitability of a particular LP to interior point methods. These are algorithms for which the iterates move within the interior of the feasible region, hence allowing for the possibility of shortcutting what would otherwise be long sequences of pivots. The addition of such measures make way for a better informed decision about which solution technique to use and whether to solve the primal or dual formulation of the problem at hand. It would be interesting to determine whether these measures, once developed, would also show a preference to solving the transhipment problem via DTP rather than TP.

We have noted that some of the results reported above can be attributed to the fact that the constraint matrix for the transhipment problem is pre-Leontief. Since this is also true for generalized transhipment problems, where the flow across each edge is scaled by some positive constant, we expect to find favorable results here as well. Preliminary investigations have shown that this is indeed the case.

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