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LOWER BOUNDS FOR MAXIMUM DIAMETERS OF POLYTOPES*

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The maximum diameter over all d-dimensional polytopes with n facets, $\Delta(d,n)$, represents the number of iterations required to solve the "worst" linear program using the ideal vertex-following algorithm. Hence $\Delta(d,n)$ measures, in a sense, the theoretical efficiency of such algorithms.

The main result of the paper is that $\Delta(d,n) \geq [(n - d) - (n - d)/[5d/4]] + 1$ for $n \geq d + 1$, and that these bounds are sharp for all known values of $\Delta(d,n)$.

0. Introduction

The diameter of a given polytope P is defined as the smallest integer k such that any two vertices of P can be joined by a path (of adjacent vertices) of length less than or equal to k. Let us denote by $\Delta(d, n)$ the maximum diameter of all d-dimensional polytopes with n facets.

The main result of this paper is the presentation of improved lower bounds for $\Delta(d, n)$.

The investigation of maximum diameters of polytopes is closely related to the study of efficiency of "vertex following" algorithms of linear programming, which start with a vertex and proceeds along successive adjacent vertices, according to some specified rule, until an optimal vertex is reached. Since, the maximum diameter of d-dimensional polytopes with n facets represents, in a sense, the number of iterations required to solve the "worst" linear program with $n - m$ equations in n

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nonnegative variables using the "best" vertex following algorithm. The main tools which are used to establish the new lower bounds for $\Delta(d, n)$ is the construction of product and summation of simple polytopes, those constructions are introduced in Section 2 together with some preliminary theorems. Then in Section 3 we present and prove the main result of the paper, namely that

$$\Delta(d, n) \geq \left[(n-d) - \frac{(n-d)}{[5d/4]}\right] + 1 \qquad (n \geq d+1),$$

and that these bounds are sharp for all the known values of $\Delta(d, n)$. It is also shown that these new bounds are slightly better than previously known lower bounds which were presented by Klee [4] and Klee and Walkup [5].

1. Notations and definitions

A *convex polytope* (or simply a *polytope*) is a bounded nonempty intersection of a finite number of closed half spaces in a finite-dimensional real vector space. The *faces* of a polytope P are the intersections of P with its various supporting hyperplanes. Zero-, one- and $(d-1)$-dimensional faces of a d-dimensional polytope P are called, respectively, the *vertices*, *edges* and *facets* of P. Two faces are said to be *incident* if one contains the other. A d-dimensional polytope is *simple* if each of its vertices is incident to exactly d edges.

Since it was shown by Klee and Walkup [5] that $\Delta(d, n)$ can be determined by considering only *simple* polytopes, we shall restrict our attention to simple polytopes and shall denote by $\mathscr{P}(d, n)$ the set of all d-dimensional simple polytopes with n facets.

As usual, $[x]$ denotes the largest integer less than or equal to x.

2. Product and sum of polytopes

2.1. *Product of polytopes*

Let $P_i \in \mathscr{P}(d_i, n_i)$ $(i = 1,2)$. We define the *product* $P_1 \otimes P_2$ of P_1

and P_2 by

$$P_1 \otimes P_2 = \{(x_1, x_2)\colon x_i \in P_i;\ i = 1, 2\}$$

Theorem 2.1.

$$P_1 \otimes P_2 \in \mathscr{P}(d_1 + d_2, n_1 + n_2).$$

Proof. The proof follows directly from the definition.

2.2. *Sum of simple polytopes*

The following construction was suggested by Barnett [3], and its combinatorial equivalent independently by Adler [1]. The following discussion follows the one given in [3].

Let $P_i \in \mathscr{P}(d, n_i)$, $i = 1, 2$.

(1) Choose arbitrarily two vertices v_1 and v_2 from P_1 and P_2, respectively.

(2) Truncate vertices v_i producing polytopes P_i^1 with simplical facets F_i $(i = 1, 2)$ which were created by the truncation.

(3) Take a hyperplane H passing through v_1 and apply a projective transformation τ_1 which sends H to infinity. In $\tau_1(P_1^1)$, all facets meeting $\tau_1(F_1)$ will be parallel. Apply the same kind of transformation τ_2 to P_2^1.

(4) Apply an affine transformation α_1 to $\tau_1(P_1^1)$ which will produce a polytope $P_1^2 = \alpha_1[\tau_1(P_1^1)]$ in which one facet meeting $\alpha_1[\tau_1(F_1)]$ is perpendicular to it. Note that all facets meeting $\alpha_1[\tau_1(F_1)]$ will be perpendicular to it. Apply the same kind of affine transformation α_2 to $\tau_2(P_2^1)$ to produce $P_2^2 = \alpha_2[\tau_2(P_2^1)]$.

(5) Apply an affine transformation α_3 to P_1^2 which will take $\alpha_1[\tau_1(F_1)]$ onto $\alpha_2[\tau_2(F_2)]$ and leaves the faces meeting $\alpha_1[\tau_1(F_1)]$ perpendicular to it.

(6) Place P_2^2 and $\alpha_3(P_1^2)$ so that $\alpha_3[\alpha_1(\tau_1(F_1))]$ and $\alpha_2[\tau_2(F_2)]$ coincide and so that the interior of P_2^2 misses the interior of $\alpha_3(P_1^2)$.

The polytope produced by this process will be called the *sum* of P_1 and P_2 and be denoted by $P_1 \oplus P_2$. Note that $P_1 \oplus P_2$ depends on the choice of v_1 and v_2 together with the choice of the several transformation mentioned above. For simplicity, we omit this dependence from the notation.

Note that all the facets of P_i (after the transformation) which do not contain v_i $(i = 1, 2)$ are facets of $P_1 \oplus P_2$ and that the d facets of P_1 which intersect at v_1 together with the d facets of P_2 which intersect at v_2 form (after the transformations) the remaining d facets of $P_1 \oplus P_2$.

Theorem 2.2.

$$P_1 \oplus P_2 \in \mathscr{P}(d, n_1 + n_2 - d).$$

Proof. The proof follows immediately from the definition and the comment following it.

3. Lower bounds for maximum diameters of polytopes

Let P be a polytope and let $v, \bar{v}$ be vertices of P. A *path* of *length* k from v to $\bar{v}$ in P is a sequence of vertices $v = v_0, \ldots, v_k = \bar{v}$ such that v_i, v_{i+1} are neighbors $(i = 0, \ldots, k-1)$. The *distance* $\rho_P(v, \bar{v})$ between v and $\bar{v}$ in P is the length of the shortest path joining v and $\bar{v}$ in P. The *diameter* $\delta(P)$ of P is defined by

$$\delta(P) = \max\{\rho_P(v, \bar{v}) \colon v, \bar{v} \in P\}.$$

Let us define $\Delta(d, n)$ as the maximum of $\delta(P)$, where P ranges over all d-dimensional polytopes with n facets.

We shall use the following two theorems in the construction of the lower bounds for $\Delta(d, n)$.

Theorem 3.1. *Let* $P_i \in \mathscr{P}(d_i, n_i)$, $i = 1,2$. *Then*

(i) $\delta(P_1 \otimes P_2) = \delta(P_1) + \delta(P_2)$.

(ii) *If* $d_1 = d_2$, *then one can sum* P_1 *and* P_2 *such that*

$$\delta(P_1) + \delta(P_2) - 1 \leq \delta(P_1 \oplus P_2) \leq \delta(P_1) + \delta(P_2).$$

Proof. (i) Let (v_1, v_2), $(\bar{v}_1, \bar{v}_2)$ be vertices of $P_1 \otimes P_2$, where $v_i, \bar{v}_i$ are vertices of P_i $(i = 1, 2)$. Let $v_i = v_i^0, \ldots, v_i^{k_i} = \bar{v}_i$ be the shortest path from v_i to $\bar{v}_i$ in P_i $(i = 1, 2)$. Then

$$\begin{aligned}(v_1, v_2) &= (v_1^0, v_2), \ldots, (v_1^{k_1}, v_2) \\ &= (\bar{v}_1, v_2^0), \ldots, (\bar{v}_1, v_2^{k_2}) = (\bar{v}_1, \bar{v}_2)\end{aligned}$$

is a path of length $k_1 + k_2$ joining (v_1, v_2) to $(\bar{v}_1, \bar{v}_2)$ in $P_1 \otimes P_2$. Hence,

$$\rho_{P_1 \otimes P_2}((v_1, v_2), (\bar{v}_1, \bar{v}_2)) \leq \rho_{P_1}(v_1, \bar{v}_1) + \rho_{P_2}(v_2, \bar{v}_2).$$

Furthermore, if (u_1, u_2), $(\bar{u}_1, \bar{u}_2)$ is a pair of adjacent vertices in $P_1 \otimes P_2$, where $u_i, \bar{u}_i \in P_i$ $(i = 1, 2)$, then either $u_1 = \bar{u}_1$ and u_2 is adjacent to $\bar{u}_2$ in P_2, or $u_2 = \bar{u}_2$ and u_1 is adjacent to $\bar{u}_1$ in P_1. Thus

$$\rho_{P_1 \otimes P_2}((v_1, v_2), (\bar{v}_1, \bar{v}_2)) \geq \rho_{P_1}(v_1, \bar{v}_1) + \rho_{P_2}(v_2, \bar{v}_2).$$

The last two inequalities imply that

$$\rho_{P_1 \otimes P_2}((v_1, v_2), (\bar{v}_1, \bar{v}_2)) = \rho_{P_1}(v_1, \bar{v}_1) + \rho_{P_2}(v_2, \bar{v}_2).$$

So $\delta(P_1 \otimes P_2) = \delta(P_1) + \delta(P_2)$.

(ii) Let $v_i, \bar{v}_i \in P_i$ such that $\rho_{P_i}(v_i, \bar{v}_i) = \delta(P_i)$, $i = 1, 2$. Now let us sum P_1 and P_2 taking v_1 and v_2 as the two vertices which are eliminated in the summation construction.

Since $\rho(v_i, \bar{v}_i) = \delta(P_i)$, it is obvious that if v_i' is adjacent to v_i in P_i, then $\rho_{P_i}(v_i', \bar{v}_i)$ is equal to either $\delta(P_i)$ or to $\delta(P_i) - 1$, $i = 1, 2$. But every vertex in P_1 which is a neighbor of v_1 has exactly one adjacent vertex in P_2 which is a neighbor of v_2 and no other vertex of P_1 has adjacent vertex in P_2. Hence,

$$\delta(P_1) + \delta(P_2) - 1 \leq \delta(P_1 \oplus P_2) \leq \delta(P_1) + \delta(P_2).$$

Theorem 3.2. (i) $\Delta(d_1 + d_2, n_1 + n_2) \geq \Delta(d_1, n_1) + \Delta(d_2, n_2)$ *and in particular*, $\Delta(d + 1, n + 2) \geq \Delta(d, n) + 1$.

(ii) $\Delta(d, n_1 + n_2 - d) \geq \Delta(d, n_1) + \Delta(d, n_2) - 1$.

Proof. (i) Let $P_i \in \mathscr{P}(d_i, n_i)$, where $\delta(P_i) = \Delta(d_i, n_i)$. By Theorem 2.1, $P_1 \otimes P_2 \in \mathscr{P}(d_1 + d_2, n_1 + n_2)$; hence by Theorem 3.1,

$$\begin{aligned}\Delta(d_1 + d_2, n_1 + n_2) \geq \delta(P_1 \otimes P_2) &= \delta(P_1) + \delta(P_2) \\ &= \Delta(d_1, n_1) + \Delta(d_2, n_2).\end{aligned}$$

If we let $P_1 \in \mathscr{P}(1,2)$ (i.e., P_1 is constituted from two adjacent vertices) then, since $\Delta(1,2) = 1$,

$$\Delta(d+1, n+2) \geq \Delta(d,n) + 1.$$

(ii) Let $P_i \in \mathscr{P}(d, n_i)$, where $\delta(P_i) = \Delta(d, n_i)$ $(i = 1,2)$. By Theorem 2.2, $P_1 \oplus P_2 \in \mathscr{P}(d, n_1 + n_2 - d)$; hence by Theorem 3.1 (summing P_1 and P_2 as specified in this theorem),

$$\begin{aligned}\Delta(d, n_1 + n_2 - d) &\geq \delta(P_1 \oplus P_2) \geq \delta(P_1) + \delta(P_2) - 1 \\ &= \Delta(d, n_1) + \Delta(d, n_2) - 1.\end{aligned}$$

We are ready now to introduce the lower bounds for $\Delta(d,n)$.

Theorem 3.3.

$$\Delta(d,n) \geq \left[(n-d) - \frac{(n-d)}{[5d/4]}\right] + 1 \quad (n \geq d+1).$$

Proof. Let

$$Z(d,n) = \left[(n-d) - \frac{(n-d)}{[5d/4]}\right] + 1.$$

It was shown by Klee and Walkup [5] that $\Delta(d,n) \geq Z(d,n)$ for $d \leq 2$. Assume that $\Delta(d-1,n) \geq Z(d-1,n)$ for some $d-1 \geq 2$ and all $n \geq d$. By Theorem 3.2 and the induction assumption,

$$\begin{aligned}\Delta(d,n) &\geq \Delta(d-1, n-2) + 1 \geq Z(d-1, n-2) + 1 \\ &= \left[(n-d-1) - \frac{(n-d-1)}{[5(d-1)/4]}\right] + 2.\end{aligned}$$

Suppose $d \neq 0 \pmod 4$, (i.e., $d/4$ is not an integer), then

$$Z(d-1, n-2) + 1 = \left[(n-d) - \frac{(n-d)-1}{[5d/4]-1}\right] + 1.$$

Thus, since $n - d \geq 1$,

$$Z(d-1, n-2) + 1 \geq Z(d,n) \quad \text{for } n-d \leq [5d/4].$$

Therefore,

$$\Delta(d,n) \geq Z(d,n) \quad \text{for } n-d \leq [5d/4] \quad (\text{and } d \neq 0 \ (\text{mod } 4)).$$

If $d = 0\,(\text{mod } 4)$, then

$$Z(d-1, n-2) + 1 = \left[(n-d) - \frac{(n-d)-1}{[5d/4]-2}\right] + 1$$

and similarly to the previous case,

$$\Delta(d,n) \geq Z(d,n) \quad \text{for } n-d \leq [5d/4] - 1$$

$$(\text{and } d = 0\,(\text{mod } 4)).$$

Furthermore, since $d = 0$ (mod 4), by Theorem 3.2 and because $\Delta(4,9) = 5$ (see Adler and Dantzig [2]),

$$\Delta\left(d, d+\frac{5d}{4}\right) = \Delta\left(\frac{d}{4}\cdot 4, \frac{d}{4}\cdot 9\right) \geq \frac{d}{4}\Delta(4,9) = \frac{5d}{4} = Z\left(d, d+\frac{5d}{4}\right).$$

Hence,

$$\Delta(d,n) \geq Z(d,n) \quad \text{for } n-d \leq [5d/4]$$

(regardless of whether $d = 0$ (mod 4) or $d \neq 0$ (mod 4).

Assume now that $\Delta(d,n) \geq Z(d,n)$ for $n \leq n_0$ (for some $n_0 \geq d + [5d/4]$). Let $(n_0 - d) = b\,\text{mod}([5d/4])$ (i.e., $(n_0 - d) - b = k[5d/4]$ for some integer k, where $0 \leq b < [5d/4]$).

By Theorem 3.2 and the induction assumption,

$$\begin{aligned}
\Delta(d, n_0+1) &\geq \Delta(d, n_0 - b) + \Delta(d, b+1+d) - 1 \\
&\geq Z(d, n_0 - b) + Z(d, b+1+d) - 1 \\
&= \left[(n_0 - b - d) - \frac{(n_0 - b - d)}{[5d/4]}\right] + 1 \\
&\quad + \left[(b+1+d-d) - \frac{(b+1+d-d)}{[5d/4]}\right] + 1 - 1
\end{aligned}$$

$$= \left[k[5d/4] - \frac{k[5d/4]}{[5d/4]} \right] + \left[(b+1) - \frac{(b+1)}{[5d/4]} \right] + 1$$

$$= \left[(n_0 + 1 - d) - \frac{(n_0 + 1 - d)}{[5d/4]} \right] + 1$$

$$= Z(d, n_0 + 1).$$

Hence, $\Delta(d, n) \geq Z(d, n)$ for all d and n for which $\Delta(d, n)$ is defined.

Remarks

(1) The previous known lower bounds for $\Delta(d, n)$ (Klee [4]) were

$$(d - 1)[n/d] - d + 2.$$

It is easily seen that the new bounds presented in Theorem 4.1 are slightly better since

$$(d-1)\left[\frac{n}{d}\right] - d + 2 \leq \left[(n-d) - \frac{n-d}{d} \right] + 1$$

$$\leq \left[(n-d) - \frac{n-d}{[5d/4]} \right] + 1.$$

In fact, Klee and Walkup [5] showed that $\Delta(4, 9) = 5$ while the old lower bound for $\Delta(4, d)$ is $(4 - 1)[9/4] - 4 + 2 = 4$. Based on this value for $\Delta(4, 9)$, Klee and Walkup [5] introduce a table of lower bounds for $\Delta(d, n)$ for $d \leq 12$ and $n \leq 24 + 2d$. It can be checked that the new lower bounds given in Theorem 4.1 are slightly better then those given in this table.

(2) The new lower bounds for $\Delta(d, n)$ are sharp for all known values of $\Delta(d, n)$ (i.e., for $d = 1, 2, 3$ and for all n and d such that $n - d \leq 5$, see [5]).

(3) Purely combinatorial proofs and discussion for the bounds established in Theorem 3.3 are given via the construction of *Abstract Polytopes* in [1] and [2].

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