Conditions for arbitrage in investment selections

Received: 21st December, 2001

Ilan Adler
is Professor in the Department of Industrial Engineering and Operations Research at the University of California, Berkeley. His areas of interest are optimization and financial engineering.

Sheldon M. Ross*
is Professor in the Department of Industrial Engineering and Operations Research at the University of California, Berkeley. He is the founding and continuing editor of the journal *Probability in the Engineering and Informational Sciences*, published by Cambridge University Press. He is a fellow of the Institute of Mathematical Statistics and a recipient of the Humboldt US senior scientist award.

*Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720, USA.
Tel: +1 510 642 3615; Fax: +1 510 642 1403; e-mail: smross@newton.me.berkeley.edu

Abstract An investment project is a vector $a = (a_0, a_1, \ldots, a_n)$ of payments over time, where a payment is made to or by the investor according to whether a_i is positive or negative. It is assumed that the investor can initiate investment a at any desired non-negative level at any period. An investment permits arbitrage if it is possible by investing suitably, without ever introducing money from the outside, to have a positive amount of money after a finite number of periods at which all investment projects have been completed. Simple sufficient conditions, in terms of linear functions on the elements of a, are provided for a given investment to permit arbitrage. The provided conditions are 'constructive' in the sense that they provide the investor with guidelines of how to achieve an arbitrage whenever one is identified. The conditions are also specialised for the more realistic case in which the investor is faced with a 'take it or leave it' decision in each period.

Keywords: *investment selection, arbitrage, cash flows valuation, positive polynomial*

Introduction

This paper deals with the situation of an agent, or investor, who is able to choose from a given set of riskless financial actions, for example, saving and/or borrowing at some given interest rate, or taking out a loan which is to be paid off over a specified number of periods, etc. In fact, it will consider the most general such action, given by any vector $\alpha = (a_0, a_1, \ldots, a_n)$, where the entry a_i is the payment or cash flow in period i. A positive cash flow is a payment to the investor, a negative flow is a payment by the investor. We call any such cash flow vector a riskless investment project. We shall assume, without loss of generality, that $a_0 \neq 0$.

Given an investment a, we shall also assume that the investor is allowed to initiate the investment at any desired non-negative level x_t at any period $t = 0, 1, \ldots$ (think of x_t as the number of 'shares' of a acquired by the investor in period t). We shall denote by $z = (z_0, z_1, \ldots, z_{n+p})$ the cash balance vector resulting from starting investment a at levels $x = (x_0, x_1, \ldots, x_p)$. By direct computation we see that:

$$z_i = \begin{cases} \sum_{j=0}^{i} s_{i-j} x_j & \text{if } i = 0, 1, \ldots, n \\ s_n \sum_{j=0}^{i-n-1} x_j + \sum_{j=i-n}^{i} s_{i-j} x_j & \\ & \text{if } i = n+1, \ldots, p \quad (1) \\ s_n \sum_{j=0}^{i-n-1} x_j + \sum_{j=i-n}^{p} s_{i-j} x_j & \\ & \text{if } i = p+1, \ldots, p+n \end{cases}$$

where $s_i = \sum_{j=0}^{i} a_j$.

Given an initial amount of money and a time horizon T (that is, the latest period in which an investment can be started), the most obvious problem for the investor is to choose investment levels (x_i in the preceding expression) so as to maximise the cash at hand in period $T + n$. The key fact being that payments to the investor from one project can be used to make payments into another while maintaining a non-negative cash balance (the z_i) at the end of each period. A more desirous goal is to achieve arbitrage. That is, by investing suitably, and without ever introducing money from the outside, to have a positive amount of money after a finite number of periods at which all investment projects have been completed. Thus,

Definition We say that a permits arbitrage if there exists a positive integer p and a non-negative $x = (x_0, x_1, \ldots, x_p)$, such that z, as defined by equation (1), satisfies $z \geq 0$ and $z_{p+n} > 0$.

Note that if an investment permits arbitrage it is possible to get 'something for nothing'. An interesting problem, which is the focus of this paper, is to be able (by applying simple calculations) to identify whether a given investment permits arbitrage when it is not trivially apparent, as in the preceding example.

We also provide an easy to calculate procedure to construct the proper investment levels needed to achieve arbitrage whenever the necessary conditions are satisfied.

In Cantor and Lippman (1983) and Adler and Gale (1997) it was proven that an investment a permits arbitrage if, and only if, the present value of a is positive for all non-negative interest rates (that is, $\sum_{i=0}^{n} a_i(1 + r)^{-i} > 0$ for all $r \geq 0$). (If negative investments are also allowed then the preceding result states that either there is an interest rate that makes all investments fair — in the present value sense — or else there is an arbitrage; this should be contrasted with the usual arbitrage theorem — see Ross (2002) — which states that either there is a probability vector that makes all bets fair or else there is an arbitrage.) In a sense, this condition makes a profitable regardless of what interest rate is used in its evaluation so we call it a 'sure bet' project. On the other hand, Pratt (1979) and Adler and Ross (2001) presented simple to calculate sufficient conditions for a given polynomial to be positive in the unit interval. Combining these results lead to sufficient conditions for arbitrage in terms of simple linear functions of the elements of a. These conditions, as well as new ones, are presented in the next section by using directly the definition for arbitrage. They are then specialised for the case in which the investment decision vector x is binary, thus restricting the investor to a 'take it or leave it' decision in each period.

Sufficient conditions for arbitrage

From the definition of arbitrage it is clear that unless a_0 as well as $\sum_{i=0}^{n} a_i$ are positive no arbitrage is possible.

Moreover, given that $a_0 > 0$ and observing equation (1) it is apparent that it is possible to maintain non-negative cash balances by simply initiating in each period investment a at a sufficiently large level. Once the last investment is initiated, however, it is still required to have non-negative cash balances in the next n periods by hopefully accumulating enough cash in the previous periods. Two set of sufficient conditions that allow the construction of a cash balance that can be sustained in the last n periods are presented below.

The following sufficient conditions for arbitrage can be inferred by combining results in Cantor and Lippman (1983) and Adler and Gale (1997) (where it was proven that a project a admits arbitrage if, and only if, the present value of a is positive for all non-negative interest rates) and the results of Pratt (1979) and Adler and Ross (2001) (where it was proven that the sufficient conditions depicted in the following theorem are sufficient for a polynomial to be positive in the unit interval). While the existing proofs (as explained above) show the existence of arbitrage indirectly, the proof below is based on explicitly demonstrating how, given the sufficient conditions are satisfied, to construct an investment plan leading to arbitrage.

Theorem 1 If $s_n > 0$ and for some $m = 0, 1, \ldots$ the following conditions hold:

$$\sum_{j=0}^{i} \binom{m+j}{m} s_{i-j} \geq 0 \quad \text{for all } i = 0, \ldots, n \tag{2}$$

$$\sum_{j=0}^{n} \binom{t+j}{t} s_{n-j} \geq 0 \quad \text{for all } t = 0, \ldots, m \tag{3}$$

then a permits arbitrage.

Proof Consider $x_j = \binom{m+j}{m}$ ($j = 0, 1, \ldots, p$ for some $p \geq n$). Then by equation (1) the resulting cash balance z_i ($i = 0, 1, \ldots, n+p$) is

$$z_i = \begin{cases} \displaystyle\sum_{j=0}^{i} s_{i-j}\binom{m+j}{m} & \text{if } i = 0, \ldots, n \\ \displaystyle s_n \sum_{j=0}^{i-n-1} \binom{m+j}{m} + \sum_{j=i-n}^{i} s_{i-j}\binom{m+j}{m} & \text{if } i = n+1, \ldots, p \\ \displaystyle s_n \sum_{j=0}^{i-n-1} \binom{m+j}{m} + \sum_{j=i-n}^{p} s_{i-j}\binom{m+j}{m}. & \text{if } i = p+1, \ldots, p+n \end{cases}$$

We shall show that for sufficiently large p, $z_i \geq 0$ for all $i = 0, 1, \ldots, p+n$.

1. For $0 \leq i \leq n$, $z_i \geq 0$ by equation (2).
2. For $n+1 \leq i \leq p$, since $s_n > 0$ it is enough to prove that $\sum_{j=i-n}^{i} s_{i-j}\binom{m+j}{m} \geq 0$. We now use mathematical induction on $i+t$ to prove that

$$H(i,t) \equiv \sum_{j=i-n}^{i} s_{i-j}\binom{t+j}{t} \geq 0 \quad \text{for all } n \leq i,\ 0 \leq t \leq m$$

By equation (3), $H(i, t) \geq 0$ for ($t = 0$, $i \geq n$) and ($i = n$, $0 \leq t \leq m$), so in particular for $i + t = n$ as well.

Suppose that $H(i, t) \geq$ for $i + t = k$ ($k \geq n$) and let $i + t = k + 1$.

Then, by noting that $\binom{t+j}{t} = \binom{t+j-1}{t-1} + \binom{t+j-1}{t}$, we obtain

$$H(i,t) = H(i,t-1) + H(i-1,t) \geq 0$$

3. For $p+1 \leq i \leq p+n$,

$$\begin{aligned} z_i &\geq s_n \sum_{j=0}^{p-n} \binom{m+j}{m} + \min\{s_0, s_1, \ldots, s_n\}\binom{m+p}{m} n \\ &= s_n \binom{m+1+p-n}{m+1} + \min\{s_0, s_1, \ldots, s_n\}\binom{m+p}{m} n \end{aligned}$$

where the preceding used the identity $\binom{m+1+k}{m+1} = \sum_{j=0}^{k} \binom{m+j}{m}$ which follows by noting that $\binom{m+j}{m}$ is the number of subsets of size $m+1$ of elements $1, \ldots, m+k+1$ whose largest numbered member is $m+j+1$.

Observing that for fixed n and m,

$$\frac{\binom{m+p}{m} n}{\binom{m+1+p-n}{m+1}} = O\left(\frac{1}{p}\right)$$

and $s_n > 0$ we get that $z_i \geq 0$ for sufficiently large p. Finally, $z_{p+n} = s_n \sum_{j=0}^{p} \binom{m+j}{m} > 0$ since $s_n > 0$.

Remark Theorem 1 raises the following question. Given a, is it possible to determine in finite time that there exists no $m = 0, 1, \ldots,$ satisfying equations (2) and (3). A partial answer to the preceding question can be clearly inferred from observing (2) and (3), namely that if there exists a positive integer t such that (2) is not satisfied for any $m \leq t$ and $\sum_{j=0}^{n} \binom{t+j}{t} s_{n-j} < 0$ then there exist no m satisfying (2) and (3).

Next, we present new sufficient conditions for arbitrage.

Theorem 2 If $s_n > 0$ and there exists non-zero non-negative $w = (w_0, \ldots, w_n)$ such that

$$\sum_{j=0}^{i} s_{i-j} w_j \geq 0 \qquad \text{for all } i = 0, \ldots, n \qquad (4)$$

and

$$\begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_n \end{pmatrix} \equiv \begin{pmatrix} s_n + s_0 & s_n & \cdots & s_2 & s_1 \\ s_n + s_1 & s_n + s_0 & \cdots & s_3 & s_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s_n + s_n & s_n + s_{n-1} & \cdots & s_n + s_1 & s_n + s_0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{pmatrix} \geq 0 \qquad (5)$$

then a permits arbitrage.

Proof Let $\sigma(j) = j \bmod n + 1$ and consider $x_j = w_{\sigma(j)}$ $(j = 0, \ldots, p)$ where $p = r(n+1)$ for some $r > 0$.

Then by (1) the resulting cash balance z_i $(i = 0, 1, \ldots, n+p)$ is

$$z_i = \begin{cases} \sum_{j=0}^{i} s_{i-j} w_j & \text{if } i = 0, 1, \ldots, n \\ s_n(k-1) \sum_{j=0}^{n} w_j + v_l & \\ \quad (\text{where } i = k(n+1) + l) & \\ & \text{if } i = n+1, \ldots, p \\ s_n(r-1) \sum_{j=0}^{n} w_j + \sum_{j=i-p-1}^{n} s_{n-j} w_j & \\ & \text{if } i = p+1, \ldots, p+n \end{cases}$$

Note that

1. For $0 \leq i \leq n$, $z_i \geq 0$ by (4).
2. For $n + 1 \leq i \leq p$, $z_i \geq 0$ by (5) and since $s_n > 0$.
3. For $p + 1 \leq i \leq p + n$, it is clear that since $s_n > 0$ and $w \neq 0$, $z_i \geq 0$ and $z_{p+n} > 0$ for sufficiently large p.

Since conditions (4) and (5) are composed of linear inequalities, it is a simple matter to determine whether a given a does or does not satisfy them. In fact, by using the well-known Farkas Lemma or any of its derivatives (see section 6.2 in Dantzig, 1963) we have,

Theorem 3 There exists no non-zero non-negative $w = (w_0, \ldots, w_n)$ satisfying (4) and (5) if, and only if, there exist non-negative $y = (y_0, \ldots, y_n)$ and $u = (u_0, \ldots, u_n)$ such that

$$\sum_{j=1}^{n} s_{j-i} y_j + (s_n + s_{j-i}) z_i + \sum_{j=0}^{i-1} s_{n-j} u_j < 0 \qquad \text{for all } i = 0, \ldots, n \qquad (6)$$

Note that most standard methods used to solve a system of linear inequalities will provide either a solution to (4) and (5) (if one exists) or alternatively, a solution to (6) which can be used as a certificate that no solution to (4) and (5) exists (see Example 1.1(c) below).

Arbitrage with binary investments

In this section we consider the case in which in each period the investor is faced with a 'take it or leave it' decision. That is, in each period the investor either initiates the cash flow a 'as is' without any scaling or not at all. Mathematically the investment decision is restricted to binary values (0 or 1) which lead to the following modified definition.

Definition We say that a permits binary arbitrage if there exists a positive integer p and a binary $x = (x_0, \ldots, x_p)$ such that z, as defined by (1) satisfies $z \geq 0$ and $z_{p+n} > 0$.

Unlike the regular case, the requirement for binary investment decisions does not allow one to maintain a non-negative cash balance in each period by initiating investment a at a sufficiently large level. Thus we can construct the following simple necessary conditions for binary arbitrage.

Theorem 4 If a admits binary arbitrage then $s_n > 0$ and there exists non-zero binary $x = (x_0, \ldots, x_n)$ such that $\Sigma_{j=0}^{i} s_{i-j}x_j \geq 0$, for all $i = 0, \ldots, n$.

Proof Suppose $(x_0, \ldots, x_p)$ is a vector demonstrating that a admits binary arbitrage. Then, considering (1) and the definition of binary arbitrage, $z_i = \Sigma_{j=0}^{i} s_{i-j}x_j \geq 0$, for all $0 \leq i \leq n$. Moreover, $z_{p+n} = s_n \Sigma_{j=0}^{p} x_j > 0$ which establishes the necessity of $s_n > 0$.

Theorem 5 If $s_n > 0$ and there exists non-zero binary $w = (w_0, \ldots, w_n)$ such that $\Sigma_{j=0}^{i} s_{i-j}\, x_j \geq 0$ for all $i = 0, \ldots, n$ and

$$\begin{pmatrix} s_n + s_0 & s_n & \cdots & s_2 & s_1 \\ s_n + s_1 & s_n + s_0 & \cdots & s_3 & s_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ s_n + s_n & s_n + s_{n-1} & \cdots & s_n + s_1 & s_n + s_0 \end{pmatrix} \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{pmatrix} \geq 0$$

then a permits arbitrage.

The proof follows directly from the proof of Theorem 2 by replacing the requirement that x is non-negative with the requirement that it is binary.

Corollary 1 If $s_n > 0$ and $\Sigma_{j=0}^{i} s_j \geq 0$ for all $i = 0, \ldots, n$ then a admits binary arbitrage.

Proof Follows directly from Theorem 5 by substituting $w_i = 1$ for $i = 0, \ldots, n$.

Note that the preceding corollary, together with Theorem 4 implies that its conditions are necessary and sufficient for obtaining arbitrage by initiating a each period within finite horizon. In the next section we demonstrate that it is possible to obtain binary arbitrage even if the conditions of Corollary 1 fail.

Examples

We present here a few simple examples to demonstrate the conditions obtained in the previous sections. Note that the proofs of the sufficient conditions for arbitrage are constructive, in the sense that whenever the conditions are satisfied, the proofs suggest easily constructed investment plans that obtains arbitrage in a finite number of periods. Moreover, any of the sufficient conditions given above, whenever satisfied, provides an assurance that the project is a 'sure bet'.

Example 1

Consider $a = (3, -6, 1, -8, a_5)$.

1. $a_5 = 21$,
   (a) a admits arbitrage (eg $x = (7,7,11,38,43,56,147,100)$ achieves arbitrage).
   (b) The conditions of Theorem 1 fail since $\Sigma_{i=0}^{5} s_{5-i} = -1$ (see the remark following Theorem 1 and consider $t = 0$).
   (c) The conditions of Theorem 2 fail (see Theorem 2 and consider $y = (0, 0, 0, 149, 67)$ and $u = (0, 27, 47, 13, 0)$).
2. $a_5 = 30$,
   (a) The conditions of Theorem 1 fail since $\Sigma_{i=0}^{5} \binom{i}{1} s_{5-i} = -3$ (see the remark following Theorem 1 and consider $t = 1$).
   (b) The conditions of Theorem 2 are satisfied (consider $w = (78, 78, 130, 443, 270)$). In fact, $x = w$ achieves arbitrage.
3. $a_5 = 37$,
   (a) The conditions of Theorem 1 are satisfied for $m = 4$. In fact, $x = (1, 4, 10, 20, 35, 56, 84)$ achieves arbitrage.
   (b) Since this example is the same as the preceding one with more cash generated in period 5, it is clear that the same w as in the preceding example satisfies the conditions of Theorem 2 and the same x achieves arbitrage.

Example 2

Consider $a = (1, 2, -4, -4, 8)$. It can be easily checked that the conditions of Theorem 4 cannot be satisfied, thus a permits no binary arbitrage, but can obtain non-binary arbitrage.

Example 3

Consider $a = (1, 3, -5, -4, 8)$. Since $\Sigma_{i=0}^{4} s_i = -1$, it is impossible to obtain binary arbitrage by initiating a at periods 1, 2, ... However, $w = (1, 0, 1, 1, 1)$ satisfies the conditions of Theorem 5, thus ensuring a binary arbitrage. In fact, w itself leads to arbitrage in five periods. That is, initiating a in periods 1, 3, 4 and 5 maintains non-negative cash balance with a profit of 12 at the end of period 5.

Remarks

In Example 1 we demonstrate that for an investment project that admits arbitrage, it is possible that both, none, or the second of the conditions depicted in Theorems 1 and 2 are satisfied. We were not able to construct an example in which the conditions of Theorem 1 are satisfied while those of Theorem 2 fail. Nonetheless we conjecture that such an example exists.

The conditions for arbitrage that were developed in Cantor and Lippman (1983) and Adler and Gale (1997) can be applied to the case in which transfer of surplus cash between consecutive periods is not allowed. In this case, the necessary and sufficient condition for a given investment project a admitting arbitrage is that the present value of a is positive for all interest rates (positive as well as negative). We note that all the conditions developed in our paper can be straightforwardly modified for this case by replacing the cash balance equation (1) with

$$z_i = \begin{cases} \Sigma_{j=0}^{i} a_{i-j} x_j & \text{if } i = 0, 1, \ldots, n \\ \Sigma_{j=i-n}^{i} a_{i-j} x_j & \text{if } i = n+1, \ldots, p \\ \Sigma_{j=i-n}^{p} a_{i-j} x_j & \text{if } i = p+1, \ldots, p+n \end{cases}$$

and then modify all the results and proofs accordingly.

Several papers (such as Pratt, 1979; Adler and Ross, 2001) developed conditions under which a given polynomial has no roots in the unit interval. While our focus is on arbitrage, it is clear that, considering Cantor and Lippman (1983) and Adler and Gale (1997), the results developed in the section 'Sufficient conditions for arbitrage', can be viewed as sufficient conditions for a given polynomial to have no roots in the unit interval. Moreover, considering the preceding remark, one can easily extend our results to develop sufficient conditions for a given polynomial to have no positive roots.

Acknowledgments

This research was supported by the National Science Foundation Grant DMI-9901053 with the University of California.

References

Adler, I. and Gale, D. (1997) 'Arbitrage and Growth Rate for Riskless Investments in a Stationary Economy', *Mathematical Finance*, 7, 73–81.

Adler, I. and Ross, S. M. (2001) 'A Probabilistic Approach to Identifying Positive Value Cash Flows', *Mathematical Scientist*, 26, 103–07.

Cantor, D. G. and Lippman, S. A. (1983) 'Investment Selection with Imperfect Capital Markets', *Econometrica*, 51, 1121–44.

Dantzig, G. B. (1963) *Linear Programming and Extensions*, Princeton University Press, Princeton.

Pratt, J. W. (1979) 'Finding How Many Roots a Polynomial Has in (0, 1) or (0, ∞)', *American Mathematical Monthly*, 86, 630–37.

Ross, S. M. (2002) *Introduction to Mathematical Finance: Options and other Topics*. 2nd edn, Cambridge University Press, London.