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MAXIMUM DIAMETER OF ABSTRACT POLYTOPES*

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A combinatorial structure called *abstract polytope* is introduced. It is shown that abstract polytopes are a subclass of pseudo-manifolds and include (combinatorially) simple convex polytopes as a special case.

The main objective is to determine the maximum diameter of abstract polytopes of dimension less than or equal to 5. Those results are relevant to the study of the efficiency of "vertex following" algorithms since the maximum diameter of d-dimensional polytopes with n facets represent, in a sense, the number of iterations required to solve the "worst" problem (with constraint set of d variables with n inequality constraints) using the "best" vertex following algorithm.

1. Introduction

Several algorithms of optimization problems with polytopes as constraint set are based on what might be called "vertex following" methods. These algorithms are based on the identification of a special class of feasible solutions (called vertices) and the determination of an adjacency relation among them. A vertex following algorithm starts with a vertex and proceeds along successive adjacent vertices, according to some specified rule, until an optimal vertex is reached (or until it is shown that no optimal vertex exists). The simplex algorithm of linear programming or Lemke's algorithm for the linear complementarity problem can serve as examples.

Related to the efficiency of such vertex following algorithms is the

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study of maximum diameter of polytopes. Loosely speaking, the maximum diameter of a d-dimensional polytope with n facets represents the number of iterations required to solve the "worst" problem (whose constraint set has $n - d$ equations and n nonnegative variables) in using the "best" vertex following algorithm.

In this paper, we introduce a general (and convenient) framework for investigating the structure of the vertex set of a polytope and its adjacency relations. This framework is provided by a set of three axioms which define *abstract polytopes.*

Our main objective is to establish values and bounds for the maximum diameter of abstract polytopes of dimension less than or equal to 5. These results are similar to those obtained by Klee and Walkup [4] for ordinary polytopes. Our results, however, apply to a more general class of combinatorial structures and imply theirs as a special case.

In Section 2, we introduce the three axioms defining abstract polytopes together with some of the terminology to be used in the paper.

In Section 3, we discuss the relations between abstract and ordinary polytopes.

The fourth section is intended primarily for readers who are familiar with the combinatorial topology terminology. In this section, we study the close relations between abstract polytopes and pseudo-manifolds and provide a preview of our results in terms of the later.

In Section 5, we present some preliminary results which are used in the proofs of the key theorems of Section 6. Finally, in Section 7, we summarize our results with respect to maximum diameter of abstract polytopes.

2. Abstract polytope—definition and notation

Given a finite set T of symbols, a family P of subsets of T (called vertices) forms a *d-dimensional abstract polytope* if the following three axioms are satisfied:

(i) Every vertex of P has cardinality d.

(ii) Any subset of $d - 1$ symbols of T is either contained in no vertices of P or in exactly two (called *neighbors* or *adjacent*).

(iii) Given any pair of vertices $v, \bar{v} \in P$, there exists a sequence of vertices $v = v_0, \ldots, v_k = \bar{v}$ such that

(a) v_i, v_{i+1} are neighbors $(i = 0, \ldots, k - 1)$,

(b) $\{v \cap \bar{v}\} \subset v_i, (i = 0, \ldots, k)$.

It is convenient to delete from T all symbols that are not used to define vertices. Hence, we denote by $\bigcup P$ the set of all symbols which appear in at least one vertex (i.e., $\bigcup P = \{t : t \in v$ for some $v \in P\}$).

Let u be a subset of $\bigcup P$ such that $|u| = k$, ($|u|$ denotes the cardinality of u). If $P' = \{v \in P : v \supset u\}$ is nonempty, we say that P' is the *face* of P which is generated by u, and denote it by $F_P(u)$ or simply $F(u)$ if the abstract polytope P is clear. It is not difficult to verify that the family $\{v - u : v \in F_P(u)\}$ of subsets obtained by deleting u from each vertex of such a face is a $(d - k)$- dimensional abstract polytope. In the sequel, we shall use this property of faces extensively. Whenever we refer to the abstract polytope associated with a face, it is understood that the deleting of common symbols has been performed. Since $F_P(u)$ corresponds to a $(d - k)$-dimensional abstract polytope, we say that it is a $(d - k)$-dimensional face of P. Zero, one and $(d - 1)$-dimensional faces are called, respectively, *vertices*, *edges*, and *facets*. Let P have n facets.

A d-dimensional abstract polytope with n facets is called an *(n, d)-abstract polytope.* (Note that $n = |\bigcup P|$.) We denote by $\mathscr{P}(n, d)$ the class of all (n, d)-abstract polytopes.

The *graph $G(P)$ of an abstract polytope P* is the graph whose vertices and edges correspond 1–1 to the vertices and edges of P, respectively.

Let P be an abstract polytope and let $v, \bar{v} \in P$. A *path* of *length k* from v to $\bar{v}$ in P is a sequence of vertices $v = v_0, \ldots, v_k = \bar{v}$ such that v_i, v_{i+1} are neighbors ($i = 0, \ldots, k - 1$). (Note that vertices of the path are not required to be in $F_P(v \cap \bar{v})$.) The *distance* $\rho_P(v, \bar{v})$ between v and $\bar{v}$ in P is the length of the shortest path joining v and $\bar{v}$. The *diameter* $\delta(P)$ or δP is the smallest integer k such that any two vertices of P can be joined by a path of length less than or equal to k: $\delta(P) = \max \rho_P(v, \bar{v})$ for $v, \bar{v} \in P$. We denote by $\Delta_a(n, d)$ the maximum of $\delta(P)$ over all (n, d)-abstract polytopes. This corresponds to Klee and Walkup's $\Delta_b(n, d)$ for ordinary simple polytopes [4]. In general, of course, $\Delta_a(n, d) \geq \Delta_b(n, d)$.

As stated in the Introduction, our main objective is to establish values and bounds for $\Delta_a(n, d)$. We shall show in particular that the analog of the unsolved d-step (or Hirsch) conjecture, i.e., that $\Delta_a(n, d) = n - d$ holds for $n - d \leq 5$.

3. Relation between abstract and simple polytopes

Abstract polytopes are (combinatorially) closely related to simple polytopes. A simple polytope can be expressed as the set of solutions of a bounded and non-degenerate linear program. Suppose the latter consists of m equations in n non-negative variables whose coefficient matrix is of rank m. One can associate n symbols with the index set of the n columns of the coefficient matrix. Then the family of subsets of symbols which correspond to the non-basic columns of all the basic feasible solutions (i.e., vertices) of the linear program forms an (n, d)-abstract polytope where $d = n - m$. This is true because any feasible solution is defined uniquely by the subset of $d = n - m$ non-basic variables set to zero (axiom (i)). Given a basic feasible solution, a new basic solution can be obtained by dropping any one of the d non-basic variables. Exactly one of the basic variables can be set equal to zero in its place (under non-degeneracy and boundedness). This generates a neighboring vertex (axiom (ii)). Given any two vertices v and $\bar{v}$, then by restricting ourselves to the lowest dimensional face common to v and $\bar{v}$ (i.e., holding at zero value the subset of non-basic variables common to the two vertices), a path of neighboring vertices from v to $\bar{v}$ can be found (e.g., by using the simplex method and a suitably chosen objective function) (axiom (iii)).

Although the class of abstract polytopes includes (combinatorially) that of simple polytopes, the converse is not true. Indeed, it is well known (e.g., see [2, p. 235]) that the graph of 3-dimensional simple polytope is planar. However, the graph of the 3-dimensional abstract polytope displayed in Fig. 2 is easily shown to be non-planar. Hence no simple polytope can have the graph structure of this particular abstract polytope. See also Remark 6.10.

4. Abstract polytopes and pseudo-manifolds

In this section, we explore the very close association between abstract polytopes and pseudo-manifolds. This section is intended primarily for the readers familiar with combinatorial topology terminology to provide them with a "dictionary" relating our own terminology with that of pseudo-manifolds. Since the rest of the paper is self-contained, this section can be skipped.

Definition 4.1. A *simplicial complex* K consist of a set $\{v\}$ of vertices and a set $\{s\}$ of nonempty subsets of $\{v\}$ called *simplices* such that

(i) any set consisting of exactly one vertex is a simplex,

(ii) any nonempty subset of a simplex is a simplex.

Definition 4.2. The *dimension* of a simplex s containing $d + 1$ vertices is defined to be d and such a simplex is called a *d-simplex*. If $s' \subset s$, then s' is called a *face* of s and if s' is d'-simplex, then it is called a *d'-face* of s.

Definition 4.3. A *d-dimensional pseudo-manifold* without boundary (or simply a d-pseudo-manifold) is a simplicial complex K such that

(i) every simplex of K is a face of some d-simplex of K,

(ii) every $(d - 1)$-simplex of K is the face of exactly two d-simplices of K,

(iii) if s and s' are d-simplices of K, there is a finite sequence $s = s_1, s_2, \ldots, s_k = s'$ of d-simplices of K such that s_i and s_{i+1} have a $(d - 1)$-face in common for $i = 1, \ldots, k - 1$.

Now, let us apply the following natural correspondence between abstract polytopes and pseudo-manifolds. Given a $(d + 1)$-dimensional abstract polytope P, one can associate with it a simplicial complex K as follows: Let $\bigcup P$ (see definition in Section 2) be the set of vertices of K and let $s \subset \bigcup P$ be a simplex of K if and only if $s \subset v$ for some $v \in P$. It is easily checked out that K is in fact a d-pseudo-manifold, since axioms (i)–(ii) are identical with the first two conditions of pseudo-manifolds and axiom (iii) is stronger than the third condition. However, if we try the reverse process of associating a $(d + 1)$-dimensional abstract polytope P to a given d-pseudo-manifold K (by defining the set of the vertices of P as the set of all d-simplices of K), we might fail because axiom (iii) is not necessarily satisfied. Therefore, let us restrict ourself to a special class of pseudo-manifolds as follows: Given a simplex s in a pseudo-manifold K, the *link of s in K* is the complex composed of all simplices of K which have no vertex in common with s, but which are faces of a simplex having s as a face. We say that a d-pseudo-manifold K is *locally connected*[1] if $d = 1$ or $d \geq 2$ and for each k-simplex s of K $(k \leq d - 2)$ the link of s in K is a pseudo-manifold.

With a little patience, one can verify that the correspondence (defined above) between $(d + 1)$-dimensional abstract polytopes and d-dimensional locally connected pseudo-manifolds (LCPM) is one-to-one. Moreover, if $s' \subset \bigcup P$ generates an $(i + 1)$-dimensional face F of the abstract polytope P, then the link of s' in K (the corresponding LCPM) is the i-dimensional LCPM corresponding to F. Thus, abstract polytopes and

[1] This term was suggested by D. Walkup (private communication).

locally connected pseudo-manifolds are essentially identical combinatorial structures and one can use either of them in developing the results presented in this paper. Although the terminology of pseudo-manifolds is well established and widely used, we prefer the abstract polytope terminology because of its natural association to simple polytopes (or equivalently nondegenerate linear programs) which are our primary subject of investigation. However, we shall outline here our main results and line of proofs using the combinatorial topology terminology to provide a link between that terminology and ours.

In terms of pseudo-manifolds, our purpose is to find the maximum diameter of locally connected d-pseudo-manifolds with n vertices for $n - d + 1 \leq 5$. (The *diameter* of a pseudo-manifold K is defined as the smallest integer k such that any two simplices of K can be joined by a path of adjacent simplices of length less than or equal to k.) This result follows a similar one obtained by Klee and Walkup [4] for convex polytopes. Their main argument is that every simple 3-polytope with 6, 7 or 8 facets satisfies what they term as Property A (see [4] and Remark 6.3). But this line of proof cannot be used in our case because of the existence of a (unique) 2-dimensional locally connected pseudo-manifold with 8 vertices (see Fig. 2) which violates Property A. (In fact, this 2-pseudo-manifold corresponds to a triangulated projective plane with a handle.) However, considering the uniqueness of that counterexample and using some simple arguments, we can still prove the main result, namely, that the maximum diameter of a locally connected 4-pseudo-manifold with 10 vertices is 5. (See Theorems 6.1 and 7.1.)

5. Some preliminary results

We shall make frequent use of the following theorem.

Theorem 5.1 [1]. *Given an abstract polytope P, if two vertices v, $\bar{v}$ in P do not have a symbol (say A) in common, then there exists an "A-avoiding path" joining them; i.e., there exists a path from v to $\bar{v}$ such that no vertex along the path contains A.*

The next theorem is the analog of a result of Klee and Walkup [4]. The proof here is similar.

Theorem 5.2. *For* $k = 0, 1, 2, \ldots$

(i) $\Delta_a(n, d) \leq \Delta_a(n+k, d+k)$,

(ii) $\Delta_a(n, d) \leq \Delta_a(n+k, d)$,

(iii) $\Delta_a(n, d) \leq \Delta_a(n+2k, d+k) - k$, $\quad \Delta_a(2d, d) \geq d$,

(iv) $\Delta_a(2d, d) = \Delta_a(2d+k, d+k)$.

Proof. We shall prove (i)–(iii) for $k = 1$; the extension to $k > 1$ is trivial.

Let P be an (n, d)-abstract polytope such that $\delta(P) = \Delta_a(n, d)$.

(i) Let $A \in \bigcup P$ and let $A' \notin \bigcup P$ be a new symbol, define P' as an abstract polytope identical with P *except* the symbol A' replaces A. Define $\tilde{P}$ as a new abstract polytope with vertices $v \cup A'$ and $v' \cup A$ for all $v \in P$ and all $v' \in P'$.

It is easy to verify that $\tilde{P}$ is an $(n+1, d+1)$-abstract polytope with a diameter at least as big as $\delta(P)$, thus

$$\Delta_a(n+1, d+1) \geq \delta(\tilde{P}) \geq \delta(P) = \Delta_a(n, d).$$

This inequality is sharp since it will be shown later that $\Delta_a(6, 2) = \Delta_a(7, 3) = 3$ and $\Delta_a(2d, d) = \Delta_a(2d+k, d+k)$ for all $k \geq 0$.

(ii) Let $A' \notin \bigcup P$ be a new symbol and $v' \in P$. Let $v_1, \ldots, v_d$ be the d subsets of v' with cardinality $d-1$. Define

$$\tilde{P} = P \setminus \{v'\} \cup \{v_i^*: v_i^* = v_i \cup A';\ i = 1, \ldots, d\}.$$

It is obvious that $\tilde{P} \in \mathscr{P}(n+1, d)$ (i.e., $\tilde{P}$ is an $(n+1, d)$-abstract polytope) and that $\delta(\tilde{P}) \geq \delta(P)$, hence

$$\Delta_a(n+1, d) \geq \delta(\tilde{P}) \geq \delta(P) = \Delta_a(n, d).$$

This inequality is also sharp since it will be shown that $\Delta_a(n, 2) = \Delta_a(n+1, 2) = n/2$ for n even.

(iii) Let $A'_1, A'_2 \notin \bigcup P$ be two new distinct symbols. Define $P_i = \{(v \cup A'_i): v \in P\}$, $i = 1,2$. Then $P_1 \cup P_2 \in \mathscr{P}(n+2, d+1)$ and $\delta(P_1 \cup P_2) = \delta(P) + 1$. So

$$\Delta_a(n+2, d+1) - 1 \geq \delta(P_1 \cup P_2) - 1 = \delta(P) = \Delta_a(n, d).$$

In particular, $1 = \Delta_a(2, 1) \leq \Delta_a(2d, d) - (d-1)$ or $\Delta_a(2d, d) \geq d$.

(iv) Let $P \in \mathscr{P}(2d+k, d+k)$, $(k \geq 0)$, where $\delta(P) = \Delta_a(2d+k, d+k)$.

Choose $v, \bar{v} \in P$ so that the shortest path from v to $\bar{v}$ has length $\delta(P)$. Note that $|v \cap \bar{v}| = k + \ell, 0 \leq \ell \leq d$. Consider the face $P' = F_P(v \cap \bar{v})$ of P which corresponds 1–1 to a $(d + k' - j, k')$-abstract polytope, where $k' = d + k - |v \mathscr{P} \bar{v}| = d - \ell$ and $0 \leq j \leq k$. Since $P' \subset P$, the length of the shortest path from v to $\bar{v}$ in P' is at least as large as $\delta(P)$. Hence

$$\Delta_a(2d - \ell - j, d - \ell) \geq \delta(P') \geq \delta(P) = \Delta_a(2d + k, d + k).$$

However, by (i) and (ii),

$$\Delta_a(2d + k, d + k) \geq \Delta_a(2d, d) \geq \Delta_a(2d - \ell - j, d - \ell).$$

Hence

$$\Delta_a(2d + k, d + k) = \Delta_a(2d, d)$$

Theorem 5.3. *Given $d > 1$ and $k \geq 0$, there exists a $P \in \mathscr{P}(2d + k, d)$ with disjoint vertices $v^*, \bar{v}^*$ whose distance $\rho(v^*, \bar{v}^*)$ is $\Delta_a(2d + k, d)$.*

Proof. Let $P_1 \in \mathscr{P}(n, d)$, where $n \geq 2d$ have two vertices $v_0, \bar{v}_0$ such that $\rho_{P_1}(v_0, \bar{v}_0) = \Delta_a(n, d)$. Assume $s = |v_0 \cap \bar{v}_0| > 0$ and consider the face $P_2 = F_{P_1}(v_0 \cap \bar{v}_0) \in \mathscr{P}(n - s - j, d - s)$, where $j \geq 0$ is the number of symbols in the set $\bigcup P_1 \backslash \{v_0 \cap \bar{v}_0)$ not used to form the vertices of the face. We have

$$\Delta_a(n, d) = \delta(P_1) \leq \delta F_{P_1}(v_0 \cap \bar{v}_0) \leq \Delta_a(n - s - j, d - s).$$

But by Theorem 5.2(i),(iii), $\Delta(n, d) \geq \Delta(n - s - j, d - s)$ with strict inequality if $j > 0$. Hence we conclude $j = 0$, $P_2 \in \mathscr{P}(n - s, d - s)$, and

$$\delta P_2 = \Delta_a(n - s, d - s) = \Delta_a(n, d).$$

Moreover, the vertices $v_0' = v_0 \backslash \{v_0 \cap \bar{v}_0\}$, $\bar{v}'_0 = \bar{v}_0 \backslash \{v_0 \cap \bar{v}_0\}$ of P_2 are disjoint and

$$\rho_{P_2}(v_0', \bar{v}_0') = \Delta_a(n - s, d - s) = \Delta_a(n, d).$$

Let $u = \bigcup P_2 \backslash \{v_0' \cup \bar{v}_0'\}$. Note $|u| = (n - s) - 2(d - s)$. Note also that $|u| \geq s$ if and only if $n \geq 2d$, i.e., if $n = 2d + k$, $k \geq 0$, which is in accord with our hypothesis. Select any subset of s symbols $\{A_1, A_2, \ldots, A_s\} \subset u$. We now define a new abstract polytope P by adjoining to $\bigcup P_2$ the set of new symbols $u' = \{A'_1, A'_2, \ldots, A'_s\}$. For each $v \in P_2$ we define $\{v \cup u'\} \in P$; given any generated $v' \in P$ and any i then replacing in v', A'_i by A_i if $A'_i \in v'$ and $A_i \notin v'$ generates additional $v'' \in P$. It is not difficult to show that $P \in \mathscr{P}(n,d)$ and $\delta P_2 \leq \delta P$; moreover, $\rho_{P_2}(v'_0, \bar{v}'_0) \leq \rho_P(v^*_0, \bar{v}^*_0)$, where $v^*_0 = \{v'_0 \cup u\}$ and $\bar{v}^*_0 = \{\bar{v}'_0 \cup u'\}$ Note that $v'_0 \cap u' = \emptyset$, $\bar{v}'_0 \cap u = \emptyset$ so that $v'_0 \cap \bar{v}'_0 = \emptyset$ implies that v^*_0 and $\bar{v}^*_0$ are also disjoint. We have

$$\Delta_a(n, d) = \Delta_a(n-s, d-s) = \rho_{P_2}(v'_0, \bar{v}'_0) \leq \rho_P(v^*_0, \bar{v}^*_0) \leq \Delta_a(n, d).$$

Corollary 5.4. *There exists a* $P \in \mathscr{P}(2d-k, d)$ *with vertices* v^* *and* $\bar{v}^*$ *such that* $\{\bigcup P - v^*\}$ *and* $\{\bigcup P - \bar{v}^*\}$ *are disjoint and* $\rho(v^*, \bar{v}^*) = \Delta_a(2d-k, d)$.

Proof. The proof is along similar lines and will be omitted. Theorem 5.3 and Corollary 5.4 are the same for $k = 0$.

6. Key theorems

Theorem 6.1 will be used (in Section 7) to establish the values of $\Delta_a(n, 2)$ and the values of $\Delta_a(n, d)$ for all n, d such that $n - d \leq 5$. Note that Theorem 5.3 allows us to consider $P \in \mathscr{P}(2d, d)$, with disjoint vertex pairs $v_0, \bar{v}_0 \in P$ such that $\rho(v_0, \bar{v}_0) = \delta P = \Delta_a(2d, d)$.

We shall make frequent use of the notion of the "shell" bordering a set of vertices of an abstract polytope. Let P be an abstract polytope and let $Z \subset P$. A vertex v of P belongs to the i^{th} *shell* $N^i_P(Z)$ of Z in P if and only if i is the minimum length of all the paths in P joining v to the various vertices of Z. The 0-shell of Z is Z itself. The 1-shell of Z is the set of vertices which are adjacent to but not in Z. In general,

$$N^i_P(Z) = N^1_P\left(\bigcup_{j=0}^{i-1} N^j_P(Z)\right).$$

For simplicity, the 1-shell of Z in P will also be denoted by $N_P(Z)$ or simply $N(Z)$ if P is clear.

Theorem 6.1. *Given $P \in \mathscr{P}(2d, d)$ (i.e., P is a $(2d, d)$-abstract polytope) and $v_0, \bar{v}_0 \in P$ such that $v_0, \bar{v}_0$ partition $\bigcup P$. Let $(v_0, v_1, \ldots, v_k)$ and $(\bar{v}_0, \bar{v}_1, \ldots, \bar{v}_{\bar{k}})$ be two paths in P with the property $|v_i \cap \bar{v}_j| = i + j$, then such paths exist for*

(i) $d \geq 1, k = 0, \bar{k} = 1$,
(ii) $d \geq 2, k = 1, \bar{k} = 1$,
(iii) $d \geq 3, k = 2, \bar{k} = 1$,

where $\bar{v}_1$ is any given vertex in $N(\bar{v}_0)$,

(iv) $d \geq 4, k = 2, \bar{k} = 2$.

Proof (except part (iv) for $d \geq 5$). It is convenient to switch from using symbols $A_i \in T$ to symbols $\{1, \ldots, d; \bar{1}, \ldots, \bar{d}\}$, and let $v_0 = \{1, \ldots, d\}$, $\bar{v}_0 = \{\bar{1}, \ldots, \bar{d}\}$ partition P. The symbols $v_i, \bar{v}_j$ where used below satisfy $|v_i \cap \bar{v}_j| = i + j$ or will be shown to do so.

(i) Obvious.

(ii) Relable so that $\bar{v}_1 = \{1, \bar{1}, \ldots, \overline{d-1}\}$. Note that $\{\bar{d}\} \not\subset (v_0 \cup \bar{v}_1)$. By Theorem 5.1, there exists an $\bar{d}$-avoiding path between v_0 and $\bar{v}_1$ in $F(v_0 \cap \bar{v}_1)$. Thus, all the vertices of the path contain $v_0 \cap \bar{v}_1 = \{1\}$ but do not contain $\{\bar{d}\}$. Let v_1 be the neighbor of v_0 in this path. Since $\{\bar{d}\} \not\subset v_1$, it must contain, for $d \geq 2$, one symbol different from those in $v_0 \cup \{\bar{d}\}$. Hence, noting $\{1\} \subset v_1, |v_1 \cap \bar{v}_1| = 2$.

(iii) Let $\bar{v}_1 \in N(\bar{v}_0)$, then by (ii) there exists a vertex $v_1 \in N(v_0)$ such that $|v_1 \cap \bar{v}_1| = 2$. By relabeling, let $v_1 = \{1, \ldots, d-1, \bar{1}\}$, $\bar{v}_1 = \{1, \bar{1}, \ldots, \overline{d-1}\}$. Define $P' = F(v_1 \cap \bar{v}_1)$ and $W = N(v_0) \cap P'$. Note that W is the set of all vertices of $N(v_0)$ which contain both $\{1\}$ and $\{\bar{1}\}$.

By Theorem 5.1, there exists a $\bar{d}$-avoiding path from v_1 to $\bar{v}_1$ in P'. Let v_2 be a vertex of this path which belongs to $N_{P'}(W)$ (such vertex exists because $v_1 \in W$ while $\bar{v}_1 \notin W$ for $d \geq 3$). But v_2 contains $\{1, \bar{1}\}$ and one symbol out of $\{\bar{2}, \ldots, \overline{d-1}\}$, hence $|v_2 \cap \bar{v}_1| = 3$.

(iv) ($d = 4$) by (iii), there exists $\bar{v}_1 \in N(\bar{v}_0)$ and $v_2 \in N^2(v_0)$ such that $|v_2 \cap \bar{v}_1| = 3$. Since $d = 4$, the second axiom of abstract polytopes implies that v_2 is a neighbor of $\bar{v}_1$. Thus letting $\bar{v}_2 = v_2$ completes the proof of this case.

Part (iv) for $d = 5$ will be established via Theorem 6.2 and for $d \geq 6$ via Theorems 6.4, 6.5, 6.7 and 6.9.

Theorem 6.2. *Let $P \in \mathscr{P}(10, 5)$ and let (v_0, v_1, v_2); $(\bar{v}_0, \bar{v}_1)$ be paths in P satisfying $|v_i \cap \bar{v}_j| = i + j$. Define*

$$P' = F(v_1 \cap \bar{v}_1), \qquad W = N(v_0) \cap P', \qquad \bar{W} = N(\bar{v}_0) \cap P'.$$

Then either there exists a path of length 3 *connecting a vertex in* W *to a vertex in* $\bar{W}$ *or*

(a) $F(v_2 \cap \bar{v}_1)$ *is (by relabeling) the* 7*-vertex* 2*-dimensional abstract polytope displayed in Fig.* 1 *and also by heavy edges in Fig.* 2.

(b) $F(v_1 \cap \bar{v}_1)$ *is (by relabeling) the* 3*-dimensional abstract polytope given in Fig.* 2 *(note that the graph of* $F(v_1 \cap \bar{v}_1)$ *is non-planar).*

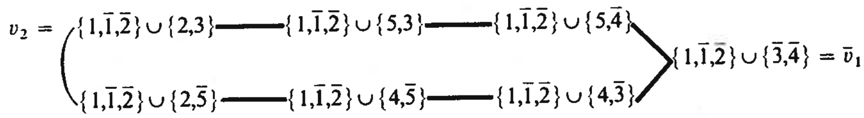


Fig. 1. Face $F(1,\bar{1},\bar{2})$.

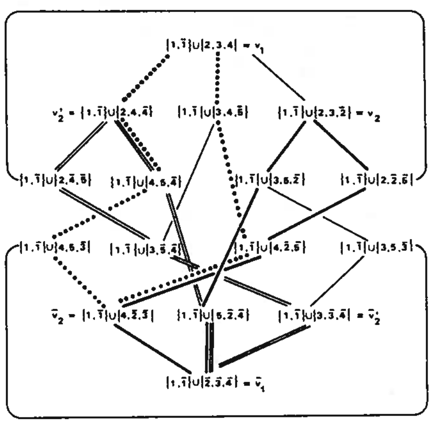


Fig. 2. Face $F(1,\bar{1})$. $F(1,\bar{1}) \in \mathscr{P}(8,3)$ has a diameter $\delta = 4$.

Proof. Assume (by relabeling if necessary) that

$$v_0 = \{1,2,3,4,5\}, \qquad v_1 = \{1,2,3,4,\bar{1}\}, \qquad v_2 = \{1,2,3,\bar{1},\bar{2}\},$$
$$\bar{v}_0 = \{\bar{1},\bar{2},\bar{3},\bar{4},\bar{5}\}, \qquad \bar{v}_1 = \{\bar{1},\bar{2},\bar{3},\bar{4},1\}.$$

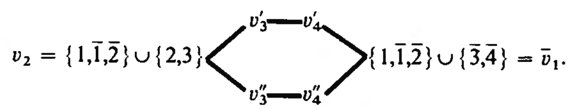
(a) Since $|v_2 \cap \bar{v}_1| = 3$, $P'' = F(v_2 \cap \bar{v})$ corresponds 1–1 to an $(n,2)$-abstract polytope. It is easy to show that every $(n,2)$-abstract polytope

Q has exactly n vertices and that every symbol of $\bigcup Q$ is contained by two adjacent vertices of Q. Furthermore, the graph of Q forms a simple cycle with diameter $[n/2]$. It is obvious that the number of vertices in P'' satisfies

$$7 = |\bigcup P| - |v_2 \cap \bar{v}_1| \geq |P''| \geq |v_2 \cup \bar{v}_1| - |v_2 \cap \bar{v}_1| = 4.$$

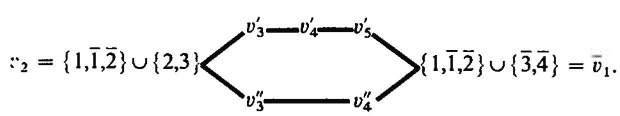
(a1) $|P''| \leq 5$. In this case $n = 4$ (or 5), there exists a path of length two joining v_2 and $\bar{v}_1$, hence of length 3 joining v_1 in W to $\bar{v}_1$ in $\bar{W}$.

(a2) $|P''| = 6$. In this case P'' has the form



Since $\{1, \bar{1}, \bar{2}\}$ is a subset of each vertex of P'', and $v_2 \cup \bar{v}_1 = \{1, \bar{1}, \bar{2}\} \cup \{2, 3, \bar{3}, \bar{4}\}$, one of the remaining symbols $\{4\}$, $\{5\}$, or $\{\bar{5}\}$ is contained by the two adjacent vertices v'_3, v'_4 and another by v''_3, v''_4. But if $\{4\}$ is contained by v'_3, v'_4 (or v''_3, v''_4), then v'_3 (or v''_3) is a neighbor of v_1. If not, then $\{\bar{5}\}$ is contained by v'_3, v'_4 (or v''_3, v''_4), and v'_4 (or v''_4) is a member of $\bar{W}$. In either case, there exists a path of length 3 from a member of W to a member of $\bar{W}$.

(a3) $|P''| = 7$. Here P'' has the form



Using the same arguments as in (a2), we see that if every path from a member of W to a member of $\bar{W}$ is *not* to have a length of 3, then v'_3, v'_4 must contain $\{\bar{5}\}$; v'_4, v'_5 must contain $\{4\}$ and v''_3, v''_4 must contain $\{5\}$. Thus P'' has the form of Fig. 1, except for possible interchange of symbols $\{2\}$ with $\{3\}$ and $\{\bar{3}\}$ with $\{\bar{4}\}$.

(b) Suppose every path in P' joining a member of W to a member of $\bar{W}$ has a length larger than 3, so that P'' as displayed in Fig. 2 by heavy lines has the form given in Fig. 1. Let us denote the vertex $\{1, \bar{1}, \bar{2}\} \cup \{4, \bar{3}\} \in \{P''\}$ by $\bar{v}_2$. We can apply now the above analysis to the face $F(v_1 \cap \bar{v}_2)$, where we permute the symbols $\{1, 2, 3, 4, 5\}$ into $\{\bar{1}, \bar{3}, \bar{2}, \bar{4}, \bar{5}\}$

and $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ into $\{1, 4, 2, 3, 5\}$. Thus $F(v_1 \cap \bar{v}_2) = F(1, 4, \bar{1})$ has the following form (with the possible interchange of $\{2\}$ with $\{3\}$ and $\{\bar{2}\}$ with $\{\bar{3}\}$):

$$v_1 = \{1,4,\bar{1}\} \cup \{2,3\} \begin{array}{l} \nearrow \{1,4,\bar{1}\} \cup \{3,\bar{5}\} - \{1,4,\bar{1}\} \cup \{\bar{2},\bar{5}\} - \{1,4,\bar{1}\} \cup \{\bar{2},\bar{3}\} \\ \searrow \{1,4,\bar{1}\} \cup \{2,\bar{4}\} - \{1,4,\bar{1}\} \cup \{5,\bar{4}\} - \{1,4,\bar{1}\} \cup \{5,\bar{3}\} \end{array} \Big) = \bar{v}_2.$$

Note that interchanging $\{\bar{2}\}$ with $\{\bar{3}\}$ is not possible since $\{1, 4, \bar{1}\} \cup \{\bar{2}, \bar{5}\}$ is forced as a neighbor of $\bar{v}_2$ because it already exists as a vertex of $F(v_2 \cap \bar{v}_1)$. The above cycle is shown by dotted edges in Fig. 2 with the possibility that the symbol $\{2\}$ is interchanged with $\{3\}$ in two of its vertices.

We now let $v_2' = \{1, \bar{1}, \bar{4}\} \cup \{2, 4\}$ and apply the argument of (a) on $F(v_2' \cap \bar{v}_1) = F(1, \bar{1}, \bar{4})$. Since $\{1, \bar{1}, \bar{4}\} \cup \{5, \bar{2}\} \in F(v_2 \cap \bar{v}_1)$ and $\{1, \bar{1}, \bar{4}\} \cup \{4, 5\} \in F(v_1 \cap \bar{v}_2)$, we get a unique form for $F(v_2' \cap \bar{v}_1)$ as follows:

$$v_2' = \Big(\begin{array}{l} \{1,\bar{1},\bar{4}\} \cup \{2,4\} - \{1,\bar{1},\bar{4}\} \cup \{4,5\} - \{1,\bar{1},\bar{4}\} \cup \{5,\bar{2}\} \searrow \\ \{1,\bar{1},\bar{4}\} \cup \{2,\bar{5}\} - \{1,\bar{1},\bar{4}\} \cup \{3,\bar{5}\} - \{1,\bar{1},\bar{4}\} \cup \{3,\bar{3}\} \nearrow \end{array} \{1,\bar{1},\bar{4}\} \cup \{\bar{2},\bar{3}\} = \bar{v}_1.$$

The above cycle is displayed in Fig. 2 by double-lined edges. If we would have interchanged $\{2\}$ with $\{3\}$ in $F(v_1 \cap \bar{v}_2)$, and defined v_2' as above, except $\{2\}$ is replaced by $\{3\}$, then $F(v_2' \cap \bar{v}_1)$ would be uniquely the the above cycle, except $\{2\}$ and $\{3\}$ would be interchanged. But then we would have $\{1, \bar{1}, \bar{4}\} \cup \{2, \bar{5}\} \in F(v_2' \cap \bar{v}_1)$ and $\{1, 4, \bar{1}\} \cup \{2, \bar{5}\} \in F(v_1 \cap \bar{v}_2)$ but $\{1, \bar{1}, \bar{2}\} \cup \{2, \bar{5}\} \in F(v_2 \cap \bar{v}_1)$. So axiom (ii) of abstract polytopes would have been violated since $\{1, 2, \bar{1}, \bar{5}\}$ would be a subset of 3 vertices of P.

Hence, we can conclude that $\{2\}$ is not interchangeable with $\{3\}$ in $F(v_1 \cap \bar{v}_2)$ (i.e., the structure of this face which is given above is the only structure which is compatible with $F(v_2 \cap \bar{v}_1)$ given in (a) and the assumption that no path of length less than 4 joined a member of W to a member of $\bar{W}$).

Finally, letting $\bar{v}_2' = \{1, 3, \bar{1}\} \cup \{\bar{3}, \bar{4}\}$, we consider $F(v_1 \cap \bar{v}_2') = F(1, 3, \bar{1})$. We note that $\{1, 3, \bar{1}\} \cup \{5, \bar{2}\}$, $\{1, 3, \bar{1}\} \cup \{2, \bar{2}\}$, $\{1, 3, \bar{1}\} \cup \{4, \bar{5}\}$ and $\{1, 3, \bar{1}\} \cup \{\bar{4}, \bar{5}\}$ are already vertices belonging to $F(v_1 \cap \bar{v}_2')$. We get applying (a) that the form of $F(v_1 \cap \bar{v}_2')$ is necessarily as follows:

$$v_1 = \{1,3,\bar{1}\} \cup \{2,4\} \begin{cases} \{1,3,\bar{1}\} \cup \{4,\bar{5}\} \text{ — } \{1,3,\bar{1}\} \cup \{\bar{4},\bar{5}\} \text{ — } \{1,3,\bar{1}\} \cup \{\bar{3},\bar{4}\} \\ \{1,3,\bar{1}\} \cup \{2,\bar{2}\} \text{ — } \{1,3,\bar{1}\} \cup \{5,\bar{2}\} \text{ — } \{1,3,\bar{1}\} \cup \{5,\bar{3}\} \end{cases} = \bar{v}_2'.$$

Collecting all the vertices of the four 2-dimensional faces considered above it is not difficult to verify that they form a 3-dimensional abstract polytope which has the structure described in Fig. 2.

Remark 6.3. Klee and Walkup [4] named the property that every path from a member of W to a member of $\bar{W}$ has a length of at most 3, as "Property A". They showed that every 3-dimensional simple polytope with 8 facets satisfies Property A. We have shown that all 3-dimensional abstract polytopes with 8 facets also satisfy Property A except one, namely the one with structure given in Fig. 2. Note, however, that this structure is non-planar. For simple polytopes this cannot happen by Steinitz's theorem [2]. Therefore, noting Theorems 5.3 and 6.1(ii), the above constitutes a new proof Klee and Walkup's theorem for $\Delta_b(2d, d)$ based on simpler assumptions, i.e., that the Hirsch d-step conjecture is true for $d \leq 5$.

Proof of Theorem 6.1 (part (iv) for $d = 5$). It is obvious that (iv) holds if and only if there exists a path of length 5 from v_0 to $\bar{v}_0$, i.e., if and only if there exists a path of length 3 from a neighbor of v_0 to a neighbor of $\bar{v}_0$.

Suppose (iv) does not hold, then by Theorem 6.2(b) every 3-dimensional face of P which is generated by a member of $N(v_0)$ and a member of $N(\bar{v}_0)$ with two symbols in common has the structure of Fig. 2 after relabeling. In particular, let $v_1 = \{1,2,3,4,\bar{1}\}$, $v_2 = \{1,2,3,\bar{1},\bar{2}\}$, $\bar{v}_1 = \{1,\bar{1},\bar{2},\bar{3},\bar{4}\}$ and $P' = F(v_1 \cap \bar{v}_1) = F(1,\bar{1})$ has the form of Fig. 2. Consider $v_0 = \{1,2,3,4,5\}$ and its incident edge generated by $\{1,3,4,5\}$. The other vertex incident to this edge cannot be $\{1,3,4,5\} \cup \{\bar{i}\}$, where $\bar{i} = \bar{1},\bar{2},\bar{3},\bar{4}$, since this would imply that there is a path of length 5 from v_0 to $\bar{v}_0$ via that edge and one of the following four vertices of P':

$$\{1,3,5,\bar{1},\bar{3}\}, \qquad \{1,4,5,\bar{1},\bar{3}\}, \qquad \{1,3,5,\bar{1},\bar{2}\}, \qquad \{1,4,5,\bar{1},\bar{4}\}.$$

Hence $\{1,3,4,5,\bar{5}\}$ is a vertex adjacent to v_0.

Similar arguments lead to the conclusion that either the set $\{1,2,4,5,\bar{5}\}$ or $\{1,2,4,5,\bar{2}\}$ is the vertex other than v_0 incident to the edge generated by $\{1,2,4,5\}$.

The same argument with respect to $\bar{v}_0$ and the edge generated by $\{\bar{1}, \bar{2}, \bar{4}, \bar{5}\}$ (because of the presence of vertices $\{1, 4, \bar{1}, \bar{2}, \bar{5}\}$, $\{1, 3, \bar{1}, \bar{4}, \bar{5}\}$, $\{1, 2, \bar{1}, \bar{2}, \bar{5}\}$, $\{1, 2, \bar{1}, \bar{4}, \bar{5}\}$ in P') implies that $\{5, \bar{1}, \bar{2}, \bar{4}, \bar{5}\} \in N_P(\bar{v}_0)$. Consider now vertices $\{1, 3, 4, 5, \bar{5}\}$ and $\{5, \bar{1}, \bar{2}, \bar{4}, \bar{5}\}$, by Theorem 6.2, in order not to have a path of length 3 joining these two vertices, the face of their intersection $F(5, \bar{5})$ must have the structure of Fig. 2. But note in Fig. 2 that the three neighbors of v_1 which are in $N^2_{P'}(v_0)$ have the property that no two are neighbors. This rules out the possibility that $\{1,2,4,5,\bar{5}\}$ is a vertex as it would lie in this face and would be a neighbor of $\{1, 3, 4, 5, \bar{5}\}$ in $N(v_0)$ instead of $N^2(v_0)$. Hence if (iv) does not hold, $\{1, 2, 4, 5, \bar{2}\} \in N(v_0)$.

Let us now consider the face $F(\{1, 2, 4, 5, \bar{2}\} \cap \bar{v}_1\}) = F(1, \bar{2})$. By Theorem 6.2, under the assumption that (iv) does not hold, $F(1, \bar{2})$ must have the structure of Fig. 2. It contains the nonempty 2-dimensional face $F(v_2 \cap \bar{v}_1) = F(1, \bar{1}, \bar{2})$. But note that $F(1, \bar{1}, \bar{2})$ also lies in $F(v_1 \cap \bar{v}_1)$ and has the 7 vertices shown connected by heavy arcs in Fig. 2. But all 2-dimensional faces with seven vertices of the abstract polytope given in Fig. 2 have the property that one of its vertices is adjacent to v_1 in P' and thus analogously one of the seven vertices of $F(1, \bar{1}, \bar{2})$ should be adjacent to $\{1, 2, 4, 5, \bar{2}\}$ in $F\{1, \bar{2}\}$ but in fact none are, a contradiction. So (iv) must hold for $d = 5$.

The last part of Theorem 6.1(iv), for $d \geq 6$, will be proved via Theorems 6.4, 6.5 and 6.7.

Theorem 6.4. *Given $P \in \mathscr{P}(2d, d)$ (where $d \geq 4$) and paths (v_0, v_1), $\bar{v}_0$ satisfying $|v_i \cap \bar{v}_j| = i + j$ and $\bar{v}'_2, \bar{v}''_2 \in N^2(\bar{v}_0)$ satisfying $|v_1 \cap \bar{v}'_2 \cap \bar{v}''_2| = 3$, then there exists a vertex $v_2 \in N^2(v_0)$ such that $|v_2 \cap \bar{v}_2| = 4$, where either $\bar{v}_2 = \bar{v}'_2$ or $\bar{v}_2 = \bar{v}''_2$.*

Proof. By relabeling for $d \geq 4$, we are assuming

$$v_0 = \{1, \dots, d\}, \qquad \bar{v}_0 = \{\bar{1}, \dots, \bar{d}\}, \qquad v_1 = \{1, \dots, d-1, \bar{1}\},$$
$$\bar{v}'_2 = \{1, 2, \bar{1}, \dots, \bar{d}\} \setminus \{\bar{i}, \bar{j}\},$$
$$\bar{v}''_2 = \{1, 2, \bar{1}, \dots, \bar{d}\} \setminus \{\bar{k}, \bar{\ell}\} \qquad (\bar{2} \leq \bar{i}, \bar{j}, \bar{k}, \bar{\ell} \leq \bar{d}),$$

where $\bar{i}, \bar{j}, \bar{k}, \bar{\ell}$ are all distinct or $\bar{i} = \bar{k}$ and $\bar{i}, \bar{j}, \bar{\ell}$ are distinct. Note that $v_1 \cap \bar{v}'_2 \cap \bar{v}''_2 = \{1, 2, \bar{1}\}$.

By Theorem 5.1, there exists an $\bar{i}$-avoiding path from v_1 to $\bar{v}'_2$ in $P' = F(v_1 \cap \bar{v}'_2)$. Let $Z = N(v_0) \cap P'$. Since $v_1 \in Z$, while $\bar{v}'_2 \notin Z$, this path intersects $N_{P'}(Z)$, say at v_2. Note that $v_2 \in N_{P'}(Z) \subset N^2_P(v_0)$.

By definition, all the vertices of $P' = F(v_1 \cap \bar{v}'_2)$ contain the symbols $\{1, 2, \bar{1}\}$, and v_2 contains also some $\{\bar{s}\} \in \{\bar{t} : \bar{t} \in \{\bar{2}, \ldots, \bar{d}\}, \bar{t} \neq \bar{i}\}$. But either $\bar{v}'_2$ or $\bar{v}''_2$ must contain $\{\bar{s}\}$. Hence either $\bar{v}_2 = \bar{v}'_2$ or $\bar{v}_2 = \bar{v}''_2$ satisfies $|v_2 \cap \bar{v}_2| = 4$.

Theorem 6.5. *Let $P \in \mathscr{P}(2d, d)$ and let $\{v_0, v_1, v_2\}$, $\{\bar{v}_0, \bar{v}_1\}$ be two paths in P such that $|v_i \cap \bar{v}_j| = i + j$. Let $W = N(v_0) \cap F(v_1 \cap \bar{v}_1)$; then if $d \geq 6$ and $|W| \geq 2$, there exists $v'_2 \in N^2(v_0)$ and $\bar{v}'_2 \in N^2(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$.*

Proof. By relabeling, let

$$v_0 = \{1, \ldots, d\}, \qquad v_1 = \{1, \ldots, d-1, \bar{1}\}, \qquad v_2 = \{1, \ldots, d-2, \bar{1}, \bar{2}\};$$
$$\bar{v}_0 = \{\bar{1}, \ldots, \bar{d}\}, \qquad \bar{v}_1 = \{1, \bar{1}, \ldots, \overline{d-1}\}.$$

Define

$$P' = F(v_1 \cap \bar{v}_1), \qquad W = N(v_0) \cap P', \qquad P'' = F(v_2 \cap \bar{v}_1),$$
$$\bar{Z} = N(\bar{v}_0) \cap P'', \qquad \bar{U}_i = \{v \in N_{P''}(\bar{Z}) : \{i\} \subset v\} \quad (i = 2, \ldots, d).$$

Theorem 6.5 is an immediate consequence of the following lemma, because the denial of the existence of such $v'_2, \bar{v}'_2$ implies by (b4) below that $|W| = 1$ whereas by hypothesis $|W| \geq 2$.

Lemma 6.6. (a) *$\bar{U}_2, \ldots, \bar{U}_d$ partitions $N_{P''}(\bar{Z})$. At least one $\bar{U}_i \neq \emptyset$, $2 \leq i \leq d-1$.*

(b) *If there exist no $v'_2 \in N^2(v_0)$ and $\bar{v}'_2 \in N^2(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$, then*

(b1) $|\bar{U}_i| = 0$ *for* $i = 2, \ldots, d-2$,
(b2) $|\bar{U}_{d-1}| = 1$,
(b3) $|\bar{U}_d| \geq d - 4$,
(b4) $|W| = 1$ *for* $d \geq 6$.

Proof. (a) Every vertex of $P'' = F(v_2 \cap \bar{v}_1) = F(1, \bar{1}, \bar{2})$ contains $\{1\}$ and thus every vertex of $\bar{Z} = N(\bar{v}_0) \cap P''$ contains exactly one unbarred symbol—namely $\{1\}$. Every vertex of $N_{P''}(\bar{Z})$ contains $\{1\}$ and exactly one other non-barred symbol. Obviously,

$$N_{P''}(\bar{Z}) = \bigcup_{i=2}^{d} \bar{U}_i \quad \text{and} \quad \bar{U}_i \cap \bar{U}_j = \emptyset \quad \text{for } i, j = (2, \ldots, d),\ i \neq j.$$

By Theorem 5.1, there exists a $\{d\}$-avoiding path joining v_2 to $\bar{v}_1$ in P''. Since $\bar{v}_1 \in \bar{Z}$ and $v_2 \notin \bar{Z}$, this path must intersect $N_{P''}(\bar{Z})$. Thus, there exists a vertex $\bar{v}'_2 \in N_{P''}(\bar{Z}) \subset N^2(\bar{v}_0)$ which does not contain $\{d\}$ implying at least one $|\bar{U}_i| \neq 0$ for $i = 2, \ldots, d-1$.

(b1) Assume $\bar{U}_{i_0} \neq \emptyset$ and $\bar{v}'_2 \in \bar{U}_{i_0}$ for some i_0, $2 \leq i_0 \leq d-2$, then $\{1, i_0, \bar{1}, \bar{2}\} \subset \bar{v}'_2$. Hence $|v_2 \cap \bar{v}'_2| = 4$. Moreover, $\bar{v}'_2 \in N_{P''}(\bar{Z}) \subset N^2(\bar{v}_0)$. This contradicts the hypothesis of (b) and we conclude $|\bar{U}_i| = 0$ for $2 \leq i \leq d-2$.

(b2) From (b1) and the discussion under (a) we conclude $|\bar{U}_{d-1}| \geq 1$. Assume now that $|\bar{U}_{d-1}| \geq 2$ and let $\bar{v}'_2$, $\bar{v}''_2 \in \bar{U}_{d-1}$. Since $\bar{v}'_2$ and $\bar{v}''_2$ both contain $\{1, d-1, \bar{1}\}$, we have $|v_1 \cap \bar{v}_2 \cap \bar{v}'_2| = 3$. Furthermore, $\bar{v}'_2, \bar{v}''_2 \in N_{P''}(\bar{Z}) \subset N^2(\bar{v}_0)$, so by Theorem 6.4 there exists $v'_2 \in N^2(v_0)$ such that either $|v'_2 \cap \bar{v}'_2| = 4$ or $|v'_2 \cap \bar{v}''_2| = 4$, contrary to hypothesis of (b). Thus we conclude that $|\bar{U}_{d-1}| = 1$.

(b3) Suppose $|\bar{Z}| = k$. Note that $k \geq 1$ because $\bar{v}_1 \in \bar{Z}$. The vertices of $\bar{Z}$ have the form, $\{1\} \cup \bar{v}_0 \setminus \{\bar{i}\}$ for $\bar{i} \in R$, where R is a subset of k indices of $\{\bar{3}, \ldots, \bar{d}\}$.

By the second axiom of abstract polytopes, the subset $\{1\} \cup v_0 \setminus \{\bar{i}, \bar{j}\}$, $(\bar{i} \in R, \bar{j} \notin R, \bar{j} \in \{\bar{3}, \ldots, \bar{d}\})$ is contained by two vertices of P''. Thus every vertex of $\bar{Z}$ gives rise to $d-2-k$ distinct vertices in $N_{P''}(\bar{Z})$. Therefore, $|N_{P''}(\bar{Z})| = k(d-2-k)$. Hence by (a), (b1) and (b2),

$$|N_{P''}(\bar{Z})| = |\bar{U}_d| + 1 = k(d-2-k).$$

The last expression implies that

$$0 < k < d-2 \quad \text{and} \quad |\bar{U}_d| \geq d-4.$$

(b4) Finally, let us assume that $d \geq 6$ and $|W| \geq 2$, and let $v'_1 \in W$ be distinct from v_1. Note that $v \in W$ is of the form $v_0 \cup \{\bar{1}\} \setminus \{i\}$, $i \neq 1$ and that v_1 is obtained by setting $i = d$ and that v'_1 is formed by setting $i = i_0$ for some $i_0 \neq 1$ or d. Thus $\{1, d, \bar{1}\} \subset v'_1$. By (b3), either there exists $v'_2 \in N^2(v_0)$ and $\bar{v}'_2 \in N^2(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$ or $|\bar{U}_d| \geq d-4 \geq 2$

for $d \geq 6$. Accordingly, let $\bar{v}_2, \bar{v}''_2 \in \bar{U}_d$. Since both $\bar{v}'_2$ and $\bar{v}''_2$ contain $\{1, d, \bar{1}\}$, we have $|v'_1 \cap \bar{v}'_2 \cap \bar{v}''_2| = 3$ which by Theorem 6.4 implies that there exists a $v'_2 \in N^2(v_0)$ such that either $|v'_2 \cap \bar{v}'_2| = 4$ or $|v'_2 \cap \bar{v}''_2| = 4$, contrary to hypotheses of part (b). We conclude for $d \geq 6$ that $|W| = 1$.

Theorem 6.7. *If $d \geq 6$ and if there exist $v'_1, v''_1 \subset N(v_0)$ and $\bar{v}_1 \subset N(\bar{v}_0)$ such that $|v'_1 \cap v''_1 \cap \bar{v}_1| = 2$, then Theorem* 6.1(iv) *holds.*

Proof. Without loss of generality we can assume that

$$v_0 = \{1, \ldots, d\}, \qquad v_1 = \{1, \ldots, d-1, \bar{1}\},$$
$$v'_1 = \{1, \ldots, d-2, d, \bar{1}\},$$
$$\bar{v}_0 = \{\bar{1}, \ldots, \bar{d}\}, \qquad \bar{v}_1 = \{1, \bar{1}, \ldots, \overline{d-1}\},$$
$$P' = F(v_1 \cap \bar{v}_1), \qquad W = P' \cap N(v_0).$$

We wish to show that if $d \geq 6$, then there exist $v'_2 \in N^2_{P'}(v_0)$ and $\bar{v}'_2 \in N^2_{P'}(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$.

By Theorem 5.1, there exists a $\{\bar{d}\}$-avoiding path from v_1 to $\bar{v}_1$ in $P' = F(v_1 \cap \bar{v}_1) = F(1, \bar{1})$. This path intersects $N_{P'}(W)$ at v_2 (say). In this case, $v_0, v_1, v_2, \bar{v}_0, \bar{v}_1$ satisfy the conditions of Theorem 6.5. Moreover, $|W| \geq 2$ since $v_1, v'_1 \subset W$ and $d \geq 6$ so that by Theorem 6.5 there exist $v'_2 \in N^2_{P'}(v_0)$ and $\bar{v}'_2 \in N^2_{P'}(\bar{v}_0)$ such that $|v'_2 \cap \bar{v}'_2| = 4$.

Corollary 6.8. *If $d \geq 6$ and if there exist adjacent $v'_1, v''_1 \subset N(v_0)$, then Theorem* 6.1(iv) *holds.*

Proof. Let $\{\bar{1}\} \subset v'_1 \cap v''_1$ and let $\{k\} \notin v'_1$ and $\{\ell\} \notin v''_1$. For $i = (2, \ldots, d)$, let $\bar{u}_i \in N(\bar{v}_0)$ such that vertex $\bar{u}_i \supset \bar{v}_0 \setminus \{\bar{i}\}$. Let $p_i = \bar{u}_i \cap v_0$. If for some $i, p_i \notin \{k, \ell\}$, then $\{\bar{u}_i \cap v'_1 \cap v''_1\} = \{\bar{1}, p_i\}$ and Theorem 6.7 applies. Otherwise, all $\bar{u}_i$ contain $\{k\}$ or $\{\ell\}$, and there exist a $\bar{u}_s$ and $\bar{u}_t$ both of which contain $\{k\}$ (say), but then $\{\bar{u}_s \cap \bar{u}_t \cap v''_1\} = \{\bar{1}, k\}$ and again Theorem 6.7 applies.

Proof of Theorem 6.1 (part (iv) for $d \geq 6$). By Theorem 6.1(iii), we can assume the existence of paths $(v_0, v_1, v_2), (\bar{v}_0, \bar{v}_1)$ such that $|v_i \cap \bar{v}_j| = i + j$. Without loss of generality we can assume that

$v_0 = \{1, \ldots, d\}, \quad v_1 = \{1, \ldots, d-1, \bar{1}\}, \quad v_2 = \{1, \ldots, d-2, \bar{1}, \bar{2}\},$
$\bar{v}_0 = \{\bar{1}, \ldots, \bar{d}\}, \quad \{\bar{v}_1 = 1, \bar{1}, \ldots, \overline{d-1}\}.$

Let us define P', P'', W, $\bar{Z}$ and $\bar{U}_i$, $(i = 2, \ldots, d)$ as in Lemma 6.6. Since we assume that $d \geq 6$, we have by Lemma 6.6 that $|\bar{U}_d| \geq 2$. Let $\bar{v}_2, \bar{v}'_2 \in \bar{U}_d$.

If $|\bar{Z}| \geq 2$, then (considering the two vertices in $\bar{Z}$ and v_1) (iv) holds by Theorem 6.7. If $|\bar{Z}| = 1$, then $\bar{Z} = \{\bar{v}_1\}$ and necessarily $\bar{v}_2, \bar{v}'_2$ have the form

$$\bar{v}_2 = \{1, d, \bar{1}, \ldots, \overline{d-1}\} \setminus \{\bar{i}_0\},$$
$$\bar{v}'_2 = \{1, d, \bar{1}, \ldots, \overline{d-1}\} \setminus \{\bar{j}_0\}$$

for some i_0, j_0, $\bar{3} \leq \bar{i}_0, \bar{j}_0 \leq \overline{d-1}$ and $\bar{i}_0 \neq \bar{j}_0$.

Let $W' = F(1) \cap N(v_0)$. Note $|W'| = d - 1$. Every $v \in W'$ contains $1, d$, except v_1. If any $v'_1 \in \{W' \setminus v_1\}$ contains $\bar{i} \notin \{\bar{i}_0, \bar{j}_0, \bar{d}\}$, then $|v'_1 \cap \bar{v}_2 \cap \bar{v}'_2| = 3$ so that Theorem 6.1(iv) follows from Theorem 6.4. If, on the contrary, all $v \in \{W' \setminus v_1\}$ contain either $\bar{d}$ or $\bar{i}_0$ or $\bar{j}_0$, then there exists a pair $v'_1, v''_1 \in \{W' \setminus v_1\}$ both of which contain $\{1, \bar{d}\}$ or $\{1, \bar{i}_0\}$ or $\{1, \bar{j}_0\}$ because $|W'v_1| = d - 2 \geq 4$ for $d \geq 6$. We may now apply Corollary 6.8.

Theorem 6.9. (i) $\Delta_a(2d+1, d) \leq \Delta_a(2d, d-1) + 1$, *for* $d \geq 2$,
(ii) $\Delta_a(2d, d) \leq \Delta_a(2d-k, d-k) + k$, *for* $k = (1, 2, 3, 4), d - k \geq 2$.

Proof. (i) Let $P \in \mathscr{P}(2d+1, d)$ such that $\delta P = \Delta_a(2d+1, d)$ and let the minimum path joining v_0 to $\bar{v}_0$ in P has length $\Delta_a(2d+1, d)$. By Theorem 5.3, we can assume $v_0 \cap \bar{v}_0 = \emptyset$ and there exists $v_1 \in N(v_0)$ such that $|v_1 \cap \bar{v}_0| = 1$, otherwise all $v \in N(v_0)$ would be neighbors and there would be no path from v_0 to $\bar{v}_0$. The result follows since $\delta[F(v_1 \cap \bar{v}_0)] \leq \Delta_a(2d, d-1)$.

(ii) Follows immediately from Theorem 6.1.

Remark 6.10. *Relations for simple polytopes.* Note that the various arguments presented apply if the phrase "simple polytope" is substituted for abstract polytope wherever it occurs, and the term $\Delta_a(n, d)$ is replaced by $\Delta_b(n, d)$ (the maximum diameter of ordinary polytopes over all d-dimensional polytopes with n facets) and therefore the various theorems and corollaries are also valid after the replacement of these terms.

7. Maximum diameters of abstract polytopes and the Hirsch conjecture

Corresponding to the Hirsch conjecture of simple polytopes is the conjecture for abstract polytopes that

$$\Delta_a(n, d) \le n - d \qquad (d > 1, n \ge d + 1).$$

Theorem 7.1 is the analog of the results of Klee and Walkup [4] for abstract polytopes (except for $\Delta_b(n, 3) = [2n/3] - 1$ for $n \ge 9$) and is mainly based on Theorem 6.1.

Theorem 7.1. *The values of $\Delta_a(n, d)$ for $n - d \le 5$, and all d are as given in Table 1. In addition, $\Delta_a(n, 2) = [n/2]$.*

Table 1
Values of $\Delta_a(n,d)$

d \ n-d	1	2	3	4	5	
1	1	×	×	×	×	
2	1	2	2	3	3	... $\Delta_a(n,2) = [n/2]$
3	1	2	3	3	4	
4	1	2	3	4	5	
≥ 5	1	2	3	4	5	

Proof. Let $P \in \mathscr{P}(n, d)$, $\delta P = \Delta_a(n, d)$. By Theorem 5.3, we can further assume for $n \ge 2d$, that there exist $v_0, \bar{v}_0 \in P$ such that $v_0 \cap \bar{v}_0 = \emptyset$ and $\rho(v_0, \bar{v}_0) = \Delta_a(n, d)$.

(a) $2d > n$. By Theorem 5.2(iv), each column of Table 1 is constant from the main diagonal downwards.

(b) $d = 2, n \ge 4$. Since P is a 2-dimensional abstract polytope, the number of vertices of P is equal to the number of its edges, therefore the graph of P forms a simple cycle with n vertices. Hence $\Delta_a(n, d) = [n/2]$.

(c) $n = 2d, d \le 5$. Applying Theorem 6.1, $\Delta_a(2d, d) = \rho(v_0, \bar{v}_0) = d$.

(d) $d = 3, n = 7$. Let $\bigcup P \setminus \{v_0 \cup \bar{v}_0\} = A$; then by Theorem 5.1 there exists an A-avoiding path between v_0 and $\bar{v}_0$. This path intersects $N^2(v_0)$ at v_2 (say). Since every vertex in $N^2(v_0)$ contains two symbols of $\{\bigcup P \setminus v_0\}$, v_2 is necessarily adjacent to $\bar{v}_0$. Hence $\Delta_a(7, 3) \le 3$. Since $\Delta_a(7, 3) \ge \Delta_a(6, 2) = 3$, by Theorem 5.2, we obtain $\Delta_a(7, 3) = 3$.

(e) $d = 3$, $n = 8$. Let $\bigcup P = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $v_0 = \{1, 2, 3\}$, $\bar{v}_0 = \{4, 5, 6\}$, where $\Delta_a(8, 3) = \rho(v_0, \bar{v}_0)$. Fig. 2 is an abstract polytope belonging to $\mathscr{P}(8, 3)$ with diameter $\delta = 4$. Therefore, $\delta(P) \geq 4$. Assume $\delta(P) > 4$, then every vertex contains either $\{7\}$ or $\{8\}$ for otherwise a vertex in $N(v_0)$ and $\bar{v}_0$ (or in $N(\bar{v}_0)$ and v_0) would both contain a symbol in common, say $\{5\}$, and we would have $\Delta_a(8, 3) = \delta(P) \leq 1 + \delta(F(5)) \leq 1 + \Delta_a(7, 2) = 4$. Thus we can assume without loss of generality $N(v_0) = \{1, 2, 7\}; \{1, 3, 7\}; \{2, 3, 8\}$ and $N(\bar{v}_0) = \{4, 5, 7\}; \{4, 6, 8\}$; and either $\{5, 6, 7\}$ or $\{5, 6, 8\}$. Consider now the cycle $F(7)$ which can contain at most seven vertices. In the first case, the shorter leg of the cycle joining $N(v_0)$ to $N(\bar{v}_0)$ provides a path of length 2. In the second case, neither $\{4, 6, 7\}$ nor $\{5, 6, 7\}$ can appear in the cycle so that it has at most six vertices and it too provides a path of length 2. Thus $4 \leq \Delta_a(8, 3) = \rho(v_0, \bar{v}_0) \leq 4$.

(f) $d = 4$, $n = 9$. Klee and Walkup [4] exhibit a $P \in \mathscr{P}(9, 4)$ with $\delta P = 5$. Thus, by Theorem 6.9 and (e), $5 \leq \Delta_a(9, 4) \leq \Delta_a(8, 3) + 1 = 5$.

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