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and
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MAXIMUM DIAMETER OF ABSTRACT POLYTOPES

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Harry Williams, *Editor*

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Foreword

This paper is one of a series by distinguished academic economists and operations analysts published by the Program Analysis Division of IDA. The series developed from a seminar program on contemporary economics. In each case the contributors presented either a survey of the present "state of the art" in their areas of interest or addressed more specialized problems of direct interest to economists. We are publishing this series to share these seminar papers with the intellectual community at large.

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I

Abstract Polytope -- Definition and Notation

Given a finite set T of symbols, a family P of subsets of T (called vertices) forms a d-<u>dimensional abstract polytope</u> if the following three axioms are satisfied:

1. Every vertex of P has cardinality d.
2. Any subset of d-1 symbols of T is either contained in no vertices of P or in exactly two (called <u>neighbors</u> or <u>adjacent</u>).
3. Given any pair of vertices $v, \bar{v} \in P$, there exists a sequence of vertices $v = v_0, \ldots, v_k = \bar{v}$ such that
   (a) v_i, v_{i+1} are neighbors $(i = 0, \ldots, k-1)$
   (b) $\{v \cap \bar{v}\} \subset v_i$, $(i = 0, \ldots, k)$.

It is convenient to delete from T all symbols that are not used to define vertices. Hence we denote $\cup P = \{\cup v \mid v \in P\}$.

Let u be a subset of $\cup P$ such that $|u| = k$, ($|u|$ denotes the cardinality of u). If $P' = \{v \in P \mid v \supset u\}$ is nonempty we say that P' is the <u>face</u> of P which is generated by u and denote it by $F_P(u)$ or simply $F(u)$ if the abstract polytope P is clear. It is not difficult to verify that the family $\{v-u \mid v \in F_P(u)\}$ of subsets obtained by deleting u from each vertex of such a face is a (d-k) -dimensional abstract polytope. In the sequel we shall use this property of faces extensively. Whenever we refer to the abstract polytope associated with a face it is understood that the deleting of common symbols has been performed. Since $F_P(u)$ corresponds to a (d-k)-dimensional abstract polytope we say that it is a (d-k)-dimensional face of P. Zero, one and (d-1)-dimensional faces are called, respectively, vertices, edges, and facets.

A d-dimensional abstract polytope with n facets is called an (n, d)-abstract polytope. (Note that $n = |\cup P|$.) We denote by $\mathbf{P}(n, d)$ the class of all (n, d)-abstract polytopes.

The graph G(P) of an abstract polytope P is the graph whose vertices and edges correspond one to one to the vertices and edges of P, respectively.

Note that Axiom 3 is satisfied by P if, and only if, the graph of every face of P is connected. If we augment P by including all subsets of the vertices of P, then Axioms 1 - 3(a) define a (d-1)-dimensional pseudomanifold (with no boundary). Thus an abstract polytope can be made to correspond to a pseudomanifold with the property that all its faces are also pseudomanifolds.

II

Relation Between Abstract and Simple Polytopes

Abstract polytopes are (combinatorially) closely related to simple polytopes. A simple polytope can be expressed as the set of solutions of a bounded and nondegenerate linear program (see Dantzig, [4]). Suppose the latter consists of m equations in n nonnegative variables whose coefficient matrix is of rank m. One can associate n symbols with the index set of the n columns of the coefficient matrix. Then the family of subsets of symbols which correspond to the nonbasic columns of all the basic feasible solutions (i.e., vertices) of the linear program forms an (n, d)-abstract polytope where d = n - m. This is true because any feasible solution is defined uniquely by the subset of d = n-m nonbasic variables set to zero (Axiom 1). Given a basic feasible solution, a new basic solution can be obtained by dropping any one of the d nonbasic variables. Exactly one of the basic variables can be set equal to zero in its place (under nondegeneracy and boundedness). This generates a neighboring vertex (Axiom 2). Given any two vertices v and $\bar{v}$, then by restricting ourselves to the lowest dimensional face common to v and $\bar{v}$ (i.e., holding at zero value the subset of nonbasic variables common to the two vertices), a path of neighboring vertices from v to $\bar{v}$ can be found (e.g., by using the simplex method and a suitably chosen objective function) (Axiom 3).

Although the class of abstract polytopes includes (combinatorially) that of simple polytopes, the converse is not true. Indeed, by a theorem of Steinitz (see [2]) the graph of every 3-dimensional simple polytope is planar. However, the graph of the 3-dimensional abstract polytope displayed in Figure 2 (on page 14) is easily shown to be nonplanar. Hence no simple polytope can have the graph structure of this particular abstract polytope. (See also the remark at the end of Section V).

III

Paths and Diameters

Let P be an abstract polytope and let $v, \bar{v} \in P$. A <u>path</u> of <u>length</u> k from v to $\bar{v}$ in P is a sequence of vertices $v = v_0, \ldots, v_k = \bar{v}$ such that v_i, v_{i+1} are neighbors $(i = 0, \ldots, k-1)$. (Note that vertices of the path are not required to be in $F_P(v \cap \bar{v})$.) The <u>distance</u> $\rho_P(v, \bar{v})$ between v and $\bar{v}$ in P is the length of the shortest path joining v and $\bar{v}$. The <u>diameter</u> $\delta(P)$ or δP is the smallest integer k such that any two vertices of P can be joined by a path of length less than or equal to k: $\delta(P) = \text{Max}\ \rho_P(v, \bar{v})$ for $v, \bar{v} \in P$. We denote by $\Delta_a(n, d)$ the maximum of $\delta(P)$ over all (n, d)-abstract polytopes. This corresponds to Klee and Walkup's $\Delta_b(n, d)$ for ordinary simple polytopes [1]. In general, of course, $\Delta_a(n,d) \geq \Delta_b(n,d)$.

Our main objective is to establish values and bounds for $\Delta_a(n,d)$. We shall show in particular that the analog of the unsolved d-step (or Hirsch) conjecture, i.e., that $\Delta_a(n,d) \leq n-d$ holds for $n-d \leq 5$, thus paralleling results of Klee and Walkup [1] for ordinary polytopes. Our arguments, however, are based on fewer axioms and imply theirs as a special case (See Section VI.)

IV

Some Preliminary Results

We shall make frequent use of the following theorem:

Theorem 1. (Adler, Dantzig, Murty [3]). Given an abstract polytope P, if two vertices v, $\bar{v}$ in P do not have a symbol (say A) in common then there exists an "A-avoiding path" joining them; i.e., there exists a path from v to $\bar{v}$ such that no vertex along the path contains A.

The next theorem is the analog of a result of Klee and Walkup in [1]. The proof here is similar.

Theorem 2. For k = 0, 1, 2, . . .

(i) $\Delta_a(n,d) \leq \Delta_a(n+k, d+k)$

(ii) $\Delta_a(n,d) \leq \Delta_a(n+k, d)$

(iii) $\Delta_a(n,d) \leq \Delta_a(n+2k, d+k) - k$; $\quad \Delta_a(2d,d) \geq d$

(iv) $\Delta_a(2d,d) = \Delta_a(2d+k, d+k)$.

Proof. We shall prove (i)-(iii) for k = 1; the extension to k > 1 is trivial.

Let P be an (n,d)-abstract polytope such that $\delta(P) = \Delta_a(n,d)$.

(i) Let $A \in \cup P$ and let $A' \notin \cup P$ be a new symbol, define P' as an abstract polytope identical with P except the symbol A' replaces A. Define $\tilde{P}$ as a new abstract polytope with vertices $v \cup A'$ and $v' \cup A$ for all $v \in P$ and all $v' \in P'$.

$v_0 = \{1, \ldots, d\}; v_1 = \{1, \ldots, d\text{-}1, \bar{1}\}; v_2 = \{\bar{1}, \ldots, d\text{-}2, \bar{1}, \bar{2}\};$

$\bar{v}_0 = \{\bar{1}, \ldots, \bar{d}\}$ and $\bar{v}_1 = \{1, \bar{1}, \ldots, \overline{d-1}\}.$

Let us define P', P", W, $\bar{Z}$ and $\bar{U}_i$, (i = 2, . . . , d) as in the preceding Lemma. Since we assume that $d \geq 6$, we have, by the Lemma, $|\bar{U}_d| \geq 2$. Let $\bar{v}_2, \bar{v}_2' \in \bar{U}_d$.

If $|\bar{Z}| \geq 2$, then (considering the two vertices in $\bar{Z}$ and v_1) (iv) holds by Theorem 8. If $|\bar{Z}| = 1$, then $\bar{Z} = \{\bar{v}_1\}$ and necessarily $\bar{v}_2, \bar{v}_2'$ have the form:

$\bar{v}_2 = \{1, d, \bar{1}, \ldots, \overline{d-1}\} - \{\bar{i}_0\}$; $\bar{v}_2' = \{1, d, \bar{1}, \ldots, \overline{d-1}\} - \{\bar{j}_0\}$

for some i_0, j_0: $\bar{3} \leq \bar{i}_0, \bar{j}_0 \leq \overline{d-1}$ and $\bar{i}_0 \neq \bar{j}_0$.

Let $W' = F(1) \cap N(v_0)$. Note $|W'| = d-1$. Every $v \in W'$ contains $\{1, d\}$ except v_1. If any $v_1' \in \{W' - v_1\}$ contains $\bar{i} \notin \{\bar{i}_0, \bar{j}_0, \bar{d}\}$, then $|v_1' \cap \bar{v}_2' \cap \bar{v}_2'| = 3$ so that Theorem 4 (iv) follows from Theorem 6. If, on the contrary, all $v \in \{W' - v_1\}$ contain either $\bar{d}$ or $\bar{i}_0$ or $\bar{j}_0$, then there exists a pair $v_1', v_1'' \in \{W' - v_1\}$ both of which contain $\{1, \bar{d}\}$ or $\{1, \bar{i}_0\}$ or $\{1, \bar{j}_0\}$ because $|W' - v_1| = d-2 \geq 4$ for $d \geq 6$. We may now apply the corollary of Theorem 8 (Corollary 2). $\square$

Theorem 9.

(i) $\Delta_a(2d+1, d) \leq \Delta_a(2d, d-1) + 1,$ $\qquad d \geq 2,$

(ii) $\Delta_a(2d, d) \leq \Delta_a(2d-k, d-k) + k,$ $\qquad k = (1, 2, 3, 4),\ d-k \geq 2.$

Proof.

(i) Let $P \in \mathbf{P}(2d+1, d)$ such that $\delta P = \Delta_a(2d+1, d)$, and let the minimal path joining v_0 to $\bar{v}_0$ in P have length $\Delta_a(2d+1, d)$. By Theorem 3 we can assume $v_0 \cap \bar{v}_0 = \emptyset$ and that there exists $v_1 \in N(v_0)$ such that $|v_1 \cap \bar{v}_0| =$ Otherwise all $v \in N(v_0)$ would be neighbors and there would be no path from v_0 to $\bar{v}_0$. The result follows since $\delta[F(v_1 \cap \bar{v}_0)] \leq \Delta_a(2d, d-1)$.

(ii) Follows immediately from Theorem 4. $\square$

Relations for Simple Polytopes. Note that the various arguments present apply if the phrase "simple polytope" is substituted for abstract polytope wherever it occurs and the term $\Delta_a(n,d)$ is replaced by $\Delta_b(n,d)$ (the maximum diameter of ordinary polytopes over all d-dimensional polytopes with n facets). Therefore, the various theorems and corollaries are als valid after the replacement of these terms.

VI

Maximum Diameters of Abstract Polytopes and the Hirsch Conjecture

Corresponding to the Hirsch conjecture of simple polytopes is the conjecture for abstract polytopes that

$$\Delta_a(n,d) \leq n - d \qquad (d > 1,\ n \geq d+1).$$

Theorem 10 below is the analog of the results of Klee and Walkup [1] for abstract polytopes (except for $\Delta_b(n,3) = [2n/3] - 1$ for $n \geq 9$) and is based mainly on Theorem 4.

<u>Theorem 10.</u> The values of $\Delta_a(n,d)$ for $n-d \leq 5$, and all d are given in the following table. In addition, $\Delta_a(n,2) = [n/2]$.

<u>Table 1</u>

VALUES OF $\Delta_a(n,d)$

d \ n-d	1	2	3	4	5	
1	1					
2	"	2	2	3	3	. . . $\Delta_a(n, 2) = [n/2]$
3	"	"	3	3	4	
4	"	"	"	4	5	
$d \geq 5$	"	"	"	"	5	

(The quotation mark indicates that each column is constant from the main diagonal downwards.)

Proof. Let $P \in \mathbf{P}(n,d)$ and $\delta P = \Delta_a(n,d)$. By Theorem 3 we can further assume for $n \geq 2d$ that there exist $v_0, \bar{v}_0 \in P$ such that $v_0 \cap \bar{v}_0 = \emptyset$ and $\rho(v_0, \bar{v}_0) = \Delta_a(n, d)$.

(a) $2d > n$: By Theorem 2 (iv) each column of Table 1 is constant from the main diagonal downwards.

(b) $d = 2$, $n \geq 4$: Since P is a 2-dimensional abstract polytope, the number of vertices of P is equal to the number of its edges; therefore, the graph of P forms a simple cycle with n vertices. Hence $\Delta_a(n,d) = [n/2]$.

(c) $n = 2d$, $d \leq 5$. Applying Theorem 4, $\Delta_a(2d, d) = \rho(v_0, \bar{v}_0) = d$.

(d) $d = 3$, $n = 7$: Let $\cup P - \{v_0 \cup \bar{v}_0\} = A$. Then by Theorem 1 there exists an A-avoiding path between v_0 and $\bar{v}_0$. This path intersects $N^2(v_0)$, say, at v_2. Since every vertex in $N^2(v_0)$ contains two symbols of $\{\cup P - v_0\}$, v_2 is necessarily adjacent to $\bar{v}_0$. Hence $\Delta_a(7,3) \leq 3$. Since $\Delta_a(7,3) \geq \Delta_a(6,2) = 3$, by Theorem 2, we obtain $\Delta_a(7,3) = 3$.

(e) $d = 3$, $n = 8$: Let $\cup P = \{1,2,3,4,5,6,7,8\}$, $v_0 = \{1,2,3\}$, and $\bar{v}_0 = \{4,5,6\}$ where $\Delta_a(8,3) = \rho(v_0, \bar{v}_0)$. Figure 2 is an abstract polytope belonging to $\mathbf{P}(8,3)$ with diameter $\delta = 4$. Therefore $\delta(P) \geq 4$. Assume $\delta(P) > 4$. Then every vertex contains either $\{7\}$ or $\{8\}$; otherwise a vertex in $N(v_0)$ and $\bar{v}_0$ (or in $N(\bar{v}_0)$ and v_0) would both contain a symbol in common, say $\{5\}$, and we would have $\Delta_a(8,3) = \delta(P) \leq 1 + \delta(F(5)) \leq 1 + \Delta_a(7,2) = 4$. Thus we can assume without loss of generality $N(v_0) = \{1,2,7\}; \{1,3,7\}; \{2,3,8\}$ and $N(\bar{v}_0) = \{4,5,7\}; \{4,6,8\}$; and either $\{5,6,7\}$ or $\{5,6,8\}$. Consider now the cycle $F(7)$ which can contain at most seven vertices. In the first case the shorter leg of the cycle joining $N(v_0)$ to $N(\bar{v}_0)$ provides a path of length 2. In the second case neither $\{4,6,7\}$ nor $\{5,6,7\}$ can appear in the cycle so that it has at most six vertices and it too provides a path of length 2. Thus $4 \leq \Delta_a(8,3) = \rho(v_0, \bar{v}_0) \leq 4$.

(f) $d = 4$, $n = 9$: Klee and Walkup in [1] exhibit a $P \in \mathbf{P}(9,4)$ with $\delta P = 5$. Thus by Theorem 9 and (e) above, $5 \leq \Delta_a(9,4) \leq \Delta_a(8,3) + 1 = 5$.

□

References

[1] Klee, Victor and D. W. Walkup, "The d-step Conjecture for Polyhedra of Dimension $d < 6$," Acta Mathematica, 117 (1967), 53-78.

[2] Grunbaum, B., Convex Polytopes, (New York: John Wiley and Sons, 1967).

[3] Adler, Ilan, George B. Dantzig, and K. Murty, Existence of x-path in Abstract Polytopes, Technical Report No. 70-1, Department of Operations Research, Stanford University (1970).

[4] Dantzig, George B., Linear Programming and Extensions, (Princeton, New Jersey: Princeton University Press, 1963).