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EXISTENCE OF A-AVOIDING PATHS IN ABSTRACT POLYTOPES*

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Theorem. *Let P be an abstract polytope and let $A \in \bigcup P$. If v, $\bar{v} \in P \backslash F_P(A)$ (i.e., if both v and $\bar{v}$ do not contain A), then there exists a path (called a A-avoiding path) joining v and $\bar{v}$ in $G(P)$ (the graph of P) such that no vertex of that path belongs to $F_P(A)$.*

Abstract polytopes include as special cases simple convex polytopes. The latter can be represented as non-degenerate bounded feasible linear programs. The analoguous theorem for convex polytopes states: If two feasible bases have a column in common, then it is possible to pass from one to the other via a sequence of adjacent feasible basis changes all of which have the column in common. The usual proof is to assign as linear objective the maximization of the variable of the common column. Then two paths exist one from each of the initial feasible bases to an optimal basis. All bases along each path have the column in common. Hence if there is a unique optimal basis, the paths may be joined together at the optimum point to form a simple connecting path.

* All definitions and notation used here are given in [1] (the preceding paper).
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It is not hard to extend the result to the non-unique case as well. For abstract polytopes, however, there is no objective function, hence a different type of proof is required.

Proof. Let P be a d-dimensional abstract polytope.

(a) $d \leq 1$. The proof is trivial.

(b) $d = 2$. By axiom (ii), $G(P)$ forms a simple cycle whose edges correspond to the facets of P. Obviously, removing the edge A from $G(P)$ cannot disconnect $G(P)$.

(c) $d \geq 3$. Let $P' = F_P(A)$ and let $v, \bar{v} \in P \backslash P'$. By axiom (iii), there exists a sequence of adjacent vertices $v = v_0, \ldots, v_k = \bar{v}$. By axiom (ii), if $v_i \in P'$, then there exists a unique vertex $\bar{v}_i$ such that $\bar{v}_i$ is a neighbor of v_i and $\bar{v}_i \notin P'$. Let

$$u_i = \begin{cases} v_i & \text{if } v_i \notin P' \\ \bar{v}_i & \text{if } v_i \in P' \end{cases} \qquad (i = 1, \ldots, k).$$

(Note that the u_i need not be distinct.) Since $|u_i \cap u_{i+1}| \geq d - 2$ $(i = 1, \ldots, k)$, $\{u_i \cap u_{i+1}\}$ generates a face of P of dimension 0 (a vertex), or 1 (an edge), or 2 (a cycle), thus there exists, by (a)–(b), an A-avoiding path p_i on $F_P(\{u_i \cap u_{i+1}\})$ joining u_i and u_{i+1} $(i = 1, \ldots, k)$. Hence $\bigcup_{i=1}^{k} p_i$ is an A-avoiding path in P joining v to $\bar{v}$.

Reference

[1] I. Adler and G.B. Dantzig, "Maximum diameter of abstract polytopes", *Mathematical Programming Study* 1 (1974) 20–40.