Direct Reduction of PPAD Linear Complementarity Problems to Bimatrix Games

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ABSTRACT

It is well known that the problem of finding a Nash equilibrium for a bimatrix game (2-NASH) can be formulated as a linear complementarity problem (LCP). In addition, 2-NASH belongs to the complexity class PPAD (Polynomial-time Parity Argument Directed). Based on the close connection between the graph associated with the Lemke algorithm, a vertex following algorithm for LCP, and the graph used to certify a problem as belonging to PPAD, it is possible to identify most LCPs processable by the Lemke algorithm (that is, problems for which the algorithm either finds a solution or provides a certificate that no solution exists) as belonging to PPAD. The discovery that 2-NASH is PPAD-complete means that every PPAD LCP can be reduced to a 2-NASH. However, the ingeniously constructed reduction (which is designed for any PPAD problem) is very complicated, so while of great theoretical significance, it is not practical for actually solving an LCP via 2-NASH, and it does not provide the potential insight that can be gained from studying the game obtained from a problem formulated as an LCP (e.g. market equilibrium). The main result of this paper is the construction of a simple explicit reduction of PPAD LCPs to symmetric 2-NASH problems. In particular, the cost matrix associated with the resulting game is constructed from the coefficient matrix of the LCP with one extra row and column (for LCPs with guaranteed solutions) or with two extra rows and columns (for the other PPAD LCPs). In addition, we show that the reduction is a bijection and discuss its implications for solving LCPs via 2-NASH and the potential for getting a deeper insight into these LCPs.

Keywords

LCP, Linear Complementarity Problem, PPAD, Bimatrix Games, Nash Equilibrium

1. INTRODUCTION

The linear complementarity problem $LCP(q, M)$ is defined as

$$\begin{aligned} &\text{find } z \\ \text{such that } Mz + q &\geq 0, \;\; z \geq 0, \\ z^{\mathsf{T}}(Mz + q) &= 0. \end{aligned}$$

The LCP is notable for its wide range of applications, from well understood and relatively easy to solve problems such as linear and convex quadratic programming problems to $\mathcal{NP}$-hard problems. A major effort in LCP theory had been the study of variants of the Lemke algorithm, a Simplex method-like vertex following algorithm. However, while the method can be applied (and guaranteed to terminate in finite number of iterations) to any LCP, it may terminate without a solution. An LCP is *Lemke-processable* if applying the Lemke algorithm on this problem results in either a solution or with a certificate that the problem has no solution. One of the major themes of LCP research over the years has been the search for classes of matrices M for which $LCP(q, M)$ is Lemke-processable for all q. Several such classes were identified (see e.g. [CPS92], [Mur88] and the references therein).

The introduction of the $\mathcal{PPAD}$ (*Polynomial-time Parity Argument Directed*) complexity class in [Pap94] provides an elegant framework for analyzing the complexity of Lemke-processable linear complementarity problems since, in general, the directed graph induced by the Lemke method for a given LCP can be used to verify the membership of the problem in $\mathcal{PPAD}$. This development is significant with respect to LCP theory since it has been shown in [MP91] that if $\mathcal{PPAD}$ is $\mathcal{NP}$-hard then $\mathcal{NP} = \text{Co}\mathcal{NP}$, lending support to the long standing informal belief that LCPs processable by the Lemke algorithm are in some way special.

What makes the class $\mathcal{PPAD}$ particularly interesting is the fact that several well known problems, such as finding a Brouwer fixed-point, were identified in [Pap94] as $\mathcal{PPAD}$-complete. The discovery, in a string of papers ([DP05], [DP05a], [CD05] and [CD05a]), that finding a Nash equilibrium of a bimatrix game (2-NASH) is $\mathcal{PPAD}$-complete, has significant consequences in the context of LCP theory. It has been known since the early days of LCP research that the 2-NASH problem can be formulated as an LCP with roughly the same size and with the coefficient matrix belonging to one of several well known classes processable by the Lemke algorithm. The fact that 2-NASH is $\mathcal{PPAD}$-

complete means that any LCP verifiable as a member in $\mathcal{PPAD}$ (including all classes that contain 2-NASH) can be reduced to a 2-NASH problem. However, the known reduction is quite complicated. It requires several stages that involve reducing the given LCP to finding an approximate Brouwer fixed point of an appropriate function, followed by reducing the latter to 3-graphical NASH (using small polymatrix games to simulate the computation of certain simple arithmetic operations), and finally, reducing the 3-graphical NASH to 2-NASH[1]. While there seems to be no a-priori reason to suspect that Lemke-processable LCPs are reducible to bimatrix games, the discovery that 2-NASH is $\mathcal{PPAD}$-complete motivated us to search for the existence of a direct simple reduction of $\mathcal{PPAD}$ LCPs to 2-NASH problems.

The main result of this paper is the introduction of a direct, simple reduction of LCPs belonging to major classes of linear complementarity problems within $\mathcal{PPAD}$ to symmetric 2-NASH problems. We begin by reviewing the LCP and the Lemke algorithm (in Section 2) and bimatrix games and Nash equilibria (in Section 3). Next, we introduce (in Section 4) the complexity class $\mathcal{PPAD}$, briefly discuss its relation to Lemke-processable LCPs, and identify a number of LCP classes as $\mathcal{PPAD}$-complete. Our main results are presented in Sections 5 and 6. In Section 5 we introduce a very simple reduction for $LCP(q, M)$, where M belongs to a class of matrices for which a solution is guaranteed to exist for all q, to a symmetric 2-NASH. In particular, we show that this class includes all such problems that had been shown in Section 2 to be Lemke-processable. The cost matrix of the resulting bimatrix game is composed of M with an extra row and column. In particular, we show that given an LCP, its solutions correspond one-to-one to the Nash equilibria which use with positive probability the pure strategy corresponding to the extra column of the cost matrix of the resulting game. In addition, we show that the Nash equilibria which do not use the pure strategy corresponding to the extra column of the cost matrix of the resulting game correspond one-to-one to the so-called 'secondary rays' associated with the reduced LCP whenever the Lemke algorithm fails to find a solution to the problem. In Section 6 we extend the reduction of the previous section to $LCP(q, M)$ for which the existence of a solution is not guaranteed. In particular, we use an augmented LCP that allows us to identify certain secondary rays associated with $LCP(q, M)$ which certify that there is no solution to the problem. The cost matrix of the resulting bimatrix game in this case is composed of M with two extra rows and columns. We note that the reductions cover a wide range of known classes of LCPs such as the class of all LCPs for which there is a unique solution or the class of all convex quadratic programming problems (formulated as LCPs). We also point out that, basically, for a large class of LCPs (not necessarily belonging to $\mathcal{PPAD}$) one can convert any LCP in this class to a 2-NASH whose set of equilibria corresponds to the possible output of the Lemke algorithm applied to the problem. So essentially, for these problems, applying any solver for 2-NASH produces whatever the Lemke algorithm does.

These reductions are particulary useful since they provide a bijection between the reducible LCPs and their corresponding 2-NASH problems. In particular, the simplicity of the reduction and its bijection property allows for the practical use of the reults of the extensive research of using "non Lemke type" 2-NASH algorithms for solving reducible LCPs (see e.g. the surveys in [vSte02],[vSte07] and the papers introduced in [vSte10]). In addition, these reductions can be applied to investigate properties of solutions to reducible LCP via known properties of the associated 2-NASH problems. they also have the potential to give new insight into problems that can be formulated as LCPs (e.g. market equilibrium models) by studying their corresponding bimatrix games. Finally, we provide some concluding remarks in Section 7. In Appendix A we present the definitions of all the matrix classes addressed in the paper.

Throughout the paper we denote by e vectors all of whose entries are 1. Given a matrix A, we denote by $A_{i\cdot}$ the i-th row of A, by $A_{\cdot j}$ the j-th column of A, and by A_{ij} the ij-th entry of A. We denote by $\boldsymbol{\mathcal{R}}^{m\times n}$, $\boldsymbol{\mathcal{R}}_{+}^{m\times n}$, and $\boldsymbol{\mathcal{R}}_{++}^{m\times n}$ the space of $m \times n$ real matrices, the space of nonnegative $m \times n$ real matrices, and the space of positive $m \times n$ real matrices, respectively. Whenever $n = 1$ we abbreviate $\boldsymbol{\mathcal{R}}^{m\times n}$ to $\boldsymbol{\mathcal{R}}^{m}$, and whenever $m = n = 1$ we abbreviate $\boldsymbol{\mathcal{R}}^{m\times n}$ to $\boldsymbol{\mathcal{R}}$.

2. LCP AND THE LEMKE ALGORITHM

Given $M \in \boldsymbol{\mathcal{R}}^{m\times m}$, $q \in \boldsymbol{\mathcal{R}}^{m}$, the *linear complementarity problem* $LCP(q, M)$ is defined as:

$$\begin{aligned} &\text{find } z \in \boldsymbol{\mathcal{R}}^{m\times m} \\ &\text{such that } Mz + q \;\geq\; 0,\;\; z \geq 0, &\text{(1a)}\\ &\qquad z^{\mathsf{T}}(Mz + q) \;=\; 0. &\text{(1b)}\end{aligned}$$

Note that (1a)–(1b) imply

$$z_i(M_{i\cdot}z + q_i) = 0, \quad i = 1, \ldots, m. \tag{1c}$$

We denote by $FEAS(q, M)$ the set of all z satisfying (1a), and by $SOL(q, M)$ the set of all z satisfying (1a) and (1b).

In this section we present the generic Lemke algorithm (the so-called Scheme I - see [CPS92], 4.4.5). Given $LCP(q, M)$ with $q \ngeq 0$ (otherwise $z = 0$ is a trivial solution), consider the polyhedral set

$$F(q, M) \triangleq \{z \in \boldsymbol{\mathcal{R}}_{+}^{m}, s \in \boldsymbol{\mathcal{R}}_{+} |\; Mz + es + q \geq 0,\; z \geq 0\}. \tag{2}$$

We assume that $F(q, M)$ is nondegenerate[2].

Definition We say that $(\bar{z}, \bar{s}) \in F(q, M)$ is an *almost complementary (AC) solution* of $F(q, M)$, if it satisfies $\bar{z}^{\mathsf{T}}(M\bar{z} + e\bar{s} + q) = 0$.

By the nondegeneracy assumption the cardinality of the support of any AC solution is either m (in which case it is a vertex (which we call *'AC vertex'*) of $F(q, M)$), or $m + 1$ (in which case it is a point on an edge (which we call *'AC edge'*) of $F(q, M)$. If an AC vertex is contained in an AC edge, we say that the vertex is an *endpoint* of the edge. If an AC edge of $F(q, M)$ is unbounded then it corresponds to a ray

[1]A clear 'bird's-eye view' description of the reduction can be found in [DGP09].

[2]If it is not, it can be made nondegenerate by applying standard techniques such as perturbation or lexicographic ordering (see [CPS92], 4.9).

(which we call *'AC ray'*) of $F(q,M)$, which can be presented as

$$\left\{ \left(\begin{array}{c} z \\ s \end{array} \right) \mid \left(\begin{array}{c} z \\ s \end{array} \right) = \left(\begin{array}{c} \bar{z} \\ \bar{s} \end{array} \right) + \left(\begin{array}{c} \bar{u} \\ \bar{r} \end{array} \right) \lambda \ \text{ for all } \lambda \geq 0 \right\} \quad (3)$$

where

$$(\bar{z},\bar{s}) \text{ is an AC vertex of } F(q,M) \text{ with } \bar{s} > 0, \quad (4a)$$

$$\bar{u} \in SOL(e\bar{r},M) \text{ for some } \bar{r} \geq 0, \text{ and with } \left(\begin{array}{c} \bar{u} \\ \bar{r} \end{array} \right) \neq 0 \quad (4b)$$

$$\bar{z}^{\mathsf{T}}(M\bar{u} + e\bar{r}) = 0, \quad (4c)$$

$$\bar{u}^{\mathsf{T}}(M\bar{z} + e\bar{s} + q) = 0. \quad (4d)$$

Note that $(\bar{z},\bar{s})$ is the (only) endpoint of the ray.

Consider the AC ray of $F(q,M)$ with endpoint $\bar{z} = 0$, $\bar{s} = -\min_{1 \leq i \leq m} q_i$ and with $\bar{u} = 0$, $\bar{r} = 1$. We call this ray the *primary ray*, and its corresponding endpoint vertex the *initial vertex*. We call any other AC ray a *secondary ray*. Note that by the nondegeneracy assumption any secondary ray has $\bar{u} \neq 0$. We denote

$$SR(q,M) \triangleq \text{ all } (\bar{z},\bar{s},\bar{u},\bar{r}) \text{ satisfying (4a)–(4d) with } \bar{u} \neq 0. \quad (5)$$

Starting with the initial vertex of $F(q,M)$, the generic Lemke algorithm traces a unique[3] finite path of adjacent AC vertices of $F(q,M)$, terminating with either a solution to $LCP(q,M)$ or with a secondary ray of $F(q,M)$. Specifically, the algorithm ends with either an AC vertex $(\bar{z}\ \bar{s})$ of $F(q,M)$ with $\bar{s} = 0$ (so $\bar{z} \in SOL(q,M)$) or with $(\bar{z},\bar{s},\bar{u},\bar{r}) \in SR(q,M)$ (that is, a secondary ray). The algorithm solves the given $LCP(q,M)$ if either it ends with $\bar{z} \in SOL(q,M)$, or if the terminal secondary ray can certify that $SOL(q,M) = \emptyset$. Whenever the Lemke algorithm solves $LCP(q,M)$ we say that $LCP(q,M)$ is *Lemke-processable*.

Ever since the introduction of the Lemke algorithm [Lem65], extensive research efforts focused on identifying classes of matrices M for which $LCP(q,M)$ is Lemke-processable for all q. In the following we discuss two major groups of matrices containing almost all known classes of matrices M for which $LCP(q,M)$ is Lemke-processable for all q.

The first group is based on the idea that if $SR(q,M) = \emptyset$ then the Lemke algorithm outputs $\bar{z} \in SOL(q,M)$. Specifically, we consider the class of regular matrices (see [CPS92], 3.9.20) as defined below.

Definition Given $M \in \boldsymbol{\mathcal{R}}^{m \times m}$ and $d \in \boldsymbol{\mathcal{R}}^{m}_{++}$, we say that M is *d-regular* if $SOL(d\tau,M) = \{0\}$ for all $\tau \in \boldsymbol{\mathcal{R}}_{+}$. We denote the class of d-regular matrices by $\mathbf{R}(d)$.

It follows that if $M \in \mathbf{R}(e)$, then $SR(q,M) = \emptyset$ for all q as (4b) with $\bar{u} \neq 0$ can not be satisfied. Recalling that the Lemke algorithm terminates in finite number of steps with either $\bar{z} \in SOL(q,M)$ or with $(\bar{z},\bar{s},\bar{u},\bar{r}) \in SR(q,M)$, we conclude that whenever $M \in \mathbf{R}(e)$, $LCP(q,M)$ is Lemke-processable for all q.

Remark It is well known that M belongs to the *strictly semimonotone* matrix class (which is denoted by $\mathbf{E}$) if and only if $SOL(q,M) = \{0\}$ for all $q \geq 0$ (see [CPS92],3.9.11). Thus, it follows that for all $d \geq 0, \mathbf{E} \subset \mathbf{R}(d)$. In addition, $\mathbf{E}$ properly includes the *strictly copositive* matrix class ($\mathbf{C}$), and the class of all matrices whose principle minors are positive ($\mathbf{P}$). Thus, we observe that $LCP(q,M)$ with M in $\mathbf{E},\mathbf{C}$ or $\mathbf{P}$ is Lemke-processable for all q.

[3]The uniqueness is due to the assumption that $F(q,M)$ is nondegenerate.

The second group includes classes of matrices for which $SR(q,M) \neq \emptyset$ implies that $SOL(q,M) = \emptyset$. Specifically, most matrix classes with this property that have been identified in the LCP literature share the following property:

$$(\bar{z},\bar{s},\bar{u},\bar{r}) \in SR(q,M) \ \Rightarrow \ \bar{r} = 0, \quad (6a)$$

$$(\bar{z},\bar{s},\bar{u},0) \in SR(q,M) \Rightarrow FEAS(q,M) = \emptyset. \quad (6b)$$

A class of matrices that satisfy (6a) is defined below.

Definitions

- We say that $M \in \boldsymbol{\mathcal{R}}^{m \times m}$ is *d-semiregular* if for all $\tau \in \boldsymbol{\mathcal{R}}_{++}$, $SOL(d\tau,M) = \{0\}$. We denote the class of d-semiregular matrices by $\mathbf{R_0}(d)$.
- We say that M belongs to class **USR** (for 'Useful Secondary Ray') if $M \in \mathbf{R_0}(e)$ and it satisfies (6b).

Remarks

1. While the term 'd-semiregular' is introduced here for the first time, the class itself has been introduced in [Gar73] under the name $\mathbf{E}^*(d)$.
2. Note that if $M \in$ **USR** then the existence of a secondary ray for $F(q,M)$ implies that $SOL(q,M) = \emptyset$. Hence, any $LCP(q,M)$ with $M \in$ **USR** is Lemke-processable for all q.
3. It is well known that M belongs to the *semimonotone* matrix class (which is denoted by $\mathbf{E_0}$) if and only if $SOL(q,M) = \{0\}$ for all $q > 0$ (see [CPS92],3.9.3). Thus, it follows that $\mathbf{E_0} \subset \mathbf{R_0}(d)$ for all $d > 0$. In addition, $\mathbf{E_0}$ properly includes the *copositive* matrix class ($\mathbf{C_0}$), and the class of all matrices whose principle minors are nonnegative ($\mathbf{P_0}$).
4. There are two well known classes of matrices, $\mathbf{L}$ and $\mathbf{Q_0} \cap \mathbf{P_0}$, which are known to be in **USR**. In particular, major matrix classes, including *Column Sufficient* (**CSU**), *Row Sufficient* (**RSU**), and *Sufficient* (**SU**), are subsets of $\mathbf{P_0} \cap \mathbf{Q_0}$, while *Copositive Plus* ($\mathbf{C_0^+}$), and *Copositive Star* ($\mathbf{C_0^*}$) are subsets of $\mathbf{L}$. Hence $LCP(q,M)$ where M belongs to any of these classes of matrices is Lemke-processable. For a discussion of these and other Lemke-processable classes see [CPS92] and [Mur88]. Figure 1 at the end of Section 4 depicts the relationship among these classes.

3. BIMATRIX GAMES

Let $A, B \in \boldsymbol{\mathcal{R}}^{m \times n}$ respectively be the cost matrices of the row and column players of a bimatrix game. A *Nash equilibrium* of this game is a pair of vectors $x \in \boldsymbol{\mathcal{R}}^{n}$, $y \in \boldsymbol{\mathcal{R}}^{m}$ (representing mixed strategies for the row and column players respectively), satisfying

$$Ay \geq e(x^{\mathsf{T}}Ay), B^{\mathsf{T}}x \geq e(x^{\mathsf{T}}By), e^{\mathsf{T}}x = e^{\mathsf{T}}y = 1, x \geq 0, y \geq 0.$$

To simplify the presentation we restrict our attention to *symmetric bimatrix games* where $A = B^\intercal$. In particular, it has been shown in the seminal paper [Nas51] that every symmetric bimatrix game has a symmetric Nash equilibrium (that is, a Nash equilibrium where $x = y$). In addition, it is well known that the Nash equilibria for any bimatrix game with cost matrices A, B (which can be assumed, without loss of generality, to be positive) can be easily extracted from the symmetric equilibria of the symmetric bimatrix game with cost matrix $\begin{pmatrix} 0 & A \\ B^\intercal & 0 \end{pmatrix}$.

Given $C \in \mathcal{R}^{n\times n}$, we denote by $SG(C)$ the symmetric bimatrix game where the row and column players' cost matrix is C. We say that $v \in \mathcal{R}^n$ is a *symmetric Nash equilibrium* of $SG(C)$ if

$$Cv \geq e(v^\intercal Cv), \tag{7a}$$
$$v \geq 0, \tag{7b}$$
$$e^\intercal v = 1. \tag{7c}$$

Note that since $v^\intercal Cv = \sum_{i=1}^m v_i(C_{i\cdot}v)$, (7a)–(7b) imply

$$v_i(C_{i\cdot}v - v^\intercal Cv) = 0, \quad i = 1, \dots, n. \tag{7d}$$

We denote by $SNE(C)$ the set of symmetric Nash equilibria of $SG(C)$. We refer to the problem of finding a symmetric Nash equilibrium for $SG(C)$ as *solving* $SG(C)$.

There are several ways of formulating the problem of finding a Nash equilibrium of a bimatrix game as a linear complementarity problem ([?],[Eav71], [MZ91], [Sav06]). Here we adopt the reduction in [Sav06], where the problem of computing a symmetric Nash equilibrium of a symmetric bimatrix game is presented as a linear complementarity problem. In particular, let C be the cost matrix of a symmetric bimatrix game. Without loss of generality we can assume (by adding a sufficiently large constant to all the entries of C) that $C > 0$. Solving $SG(C)$ with $C > 0$ can be reduced to $LCP(-e, C)$ as described in [Sav06], and presented in the following theorem.

THEOREM 1. *Suppose $C > 0$.*

(i) *Let $z \in SOL(-e, C)$. Then, $z\frac{1}{e^\intercal z} \in SNE(C)$.*

(ii) *Let $v \in SNE(C)$. Then, $v\frac{1}{v^\intercal Cv} \in SOL(-e, C)$.*

Proof. This can be easily verified by substitution. □

4. THE COMPLEXITY CLASS PPAD

The class $\mathcal{PPAD}$ (*Polynomial-time Parity Argument Directed*), which was introduced in the seminal paper [Pap94], is a class of problems which can be presented as follows:

Definition Given a directed graph with every node having in-degree and out-degree at most one described by a polynomial-time computable function $f(v)$ that outputs the predecessor and successor of a node v, and a node s (which we call the *initial source node*) with a successor but no predecessors, find a node $t \neq s$ which is either a *sink* (a node with no successor) or a *source* (a node with no predecessor), but not both. Whenever we construct such a graph for a given problem, we call it the *$\mathcal{PPAD}$ graph* associated with the problem.

Many important problems, such as the Brouwer fixed-point problem, the search versions of Smith's theorem, the Borsuk-Ulam theorem and, as previously discussed, Nash equilibrium of bimatrix game, belong to this class [Pap94]. Interestingly, the problems in $\mathcal{PPAD}$ are generally believed to not be $\mathcal{NP}$-hard since it has been shown in [MP91] that if there exists a $\mathcal{PPAD}$ problem which is $\mathcal{NP}$-hard then $\mathcal{NP} = \mathcal{C}o\mathcal{NP}$. What makes the study of this class attractive is that it has been shown that several problems within the class (such as the Brouwer fixed-point problem) are $\mathcal{PPAD}$-complete, with strong circumstantial evidence that thiese problems are not likely to have a polynomial time algorithm [HPV89].

The $\mathcal{PPAD}$ class seems to be a natural framework for analyzing the computational complexity of Lemke-processable $LCP(q, M)$, as the underlying graph of the Lemke algorithm whose nodes correspond to AC vertices and AC edges of $F(q, M)$ has a structure reminiscent of a $\mathcal{PPAD}$ graph. Indeed, in [Pap94], one of the first examples of a $\mathcal{PPAD}$ problem is an $LCP(q, M)$ where $M \in \mathbf{P}$. While it is customary in the literature of linear complementarity to discuss methods for solving $LCP(q, M)$ under the assumption that M possesses some special properties, it creates difficulties from an algorithmic complexity point of view, as verifying these properties may be by itself a hard problem (e.g. identifying a $\mathbf{P}$ matrix is $\mathcal{C}o\mathcal{NP}$ complete [Cox73]). Thus, in [Pap94], the problem at hand (which is called $\mathbf{P} - LCP$) is defined as follows. Given M, q, either find $\bar{z} \in SOL(q, M)$, or provide a certificate (with size polynomial in the size of the problem) for $M \notin \mathbf{P}$. Motivated by the discussion in [Pap94] we consider the following generic problem:

$\mathbf{Y} - LCP(q, M)$: Given $M \in \mathcal{R}^{m\times m}$, $q \in \mathcal{R}^m$ and a matrix class $\mathbf{Y}$, find one of the following:

1. $z \in SOL(q, M)$,
2. A certificate that $SOL(q, M) = \emptyset$,
3. A certificate that $M \notin \mathbf{Y}$.

It is shown in [Pap94] how to construct an associated $\mathcal{PPAD}$ graph for the problem $\mathbf{P} - LCP(q, M)$ based on the underlying graph whose nodes correspond to the AC vertices and AC edges of $F(q, M)$. While it is possible to apply the same approach to verify the $\mathcal{PPAD}$ membership of $\boldsymbol{Y} - LCP(q, M)$ (for all q) for all classes $\boldsymbol{Y}$ that had be shown in Section 2 to be Lemke-processable, we shall not pursue this approach here. Instead, our reductions of these $\boldsymbol{Y} - LCP$ to bimatrix games in the next two sections will automatically provide $\mathcal{PPAD}$ verifications for these problems.

As stated in the introduction, it has been established that the problem of finding a Nash equilibrium for a bimatrix game (2-NASH) is $\mathcal{PPAD}$-complete. Moreover, since any 2-NASH is polynomially reducible to a symmetric 2-NASH, we have that the problem of finding a symmetric Nash equilibrium for a bimatrix game, as presented in Section 3, is also $\mathcal{PPAD}$-complete. Moreover, it is shown there that this problem can be represented as an $LCP(-e, M)$ where $M > 0$. Since $M > 0$ implies that $M \in \mathbf{C}$, the class of all matrices for which $0 \neq x \in \mathcal{R}^m_+ \Rightarrow x^\intercal Mx > 0$, and since $LCP(q, M)$ with $M \in \mathbf{C}$ is in $\mathcal{PPAD}$ (as we establish in the next section), we conclude that $\mathbf{C} - LCP$ is $\mathcal{PPAD}$-complete as well. In Figure 1, we display the relationship among the classes of matrices discussed in the previous and current sections. An arrow from class $\mathbf{X}$ to class $\mathbf{Y}$ indicates that $\mathbf{X} \subset \mathbf{Y}$. So for any $\mathbf{Y}$ reachable by a directed path from $\mathbf{C}$ in Figure 1, we have that if $\mathbf{Y} - LCP$ is in $\mathcal{PPAD}$ then it is $\mathcal{PPAD}$-complete. Note that the class **USR** contains all the classes of matrices identified in this section as matrices for which their corresponding $LCP(q, M)$ belong to $\mathcal{PPAD}$ for all q.

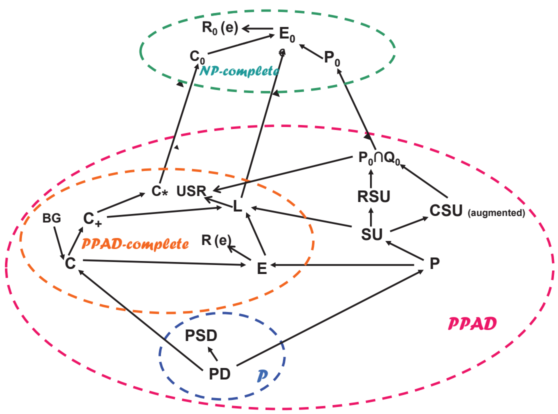

Figure 1

In the the next section we present a simple reduction from $LCP(q, M)$ with $M \in \mathbf{R}(e)$ to a symmetric NASH-2. In Section 6 we extend the reduction for $M \in \mathbf{USR}$.

5. REDUCING LCP WITH GUARANTEED SOLUTION

In this section we present a simple direct reduction of $LCP(q, M)$ where $M \in \mathbf{R}(e)$. This class, as discussed in the previous section, contains most known matrix classes of Lemke-processable LCPs which are guaranteed to have a solution for all q.

Given $LCP(q, M)$ with $M \in \mathcal{R}^{m \times m}$ and $q \in \mathcal{R}^m$, we set $n = m + 1$ and a cost matrix $C(q, M) \in \mathcal{R}^{n \times n}$ for a symmetric bimatrix game as follows:

$$C(q, M) = \begin{pmatrix} M & q + e \\ 0 & 1 \end{pmatrix}. \tag{8}$$

Given $C(q, M)$ as above we denote any symmetric equilibrium point $v \in SNE(C)$ as $v = \begin{pmatrix} u \\ t \end{pmatrix}$, where $u \in \mathcal{R}^m$ and $t \in \mathcal{R}$. Given $SNE(C(q, M))$, we partition it to

$$SNE_+(C(q, M)) \triangleq \left\{ \begin{pmatrix} u \\ t \end{pmatrix} \in SNE(C(q, M)) \mid t > 0 \right\},$$

and

$$SNE_0(C(q, M)) \triangleq \left\{ \begin{pmatrix} u \\ t \end{pmatrix} \in SNE(C(q, M)) \mid t = 0 \right\},$$

In the next theorem we establish a one-to-one correspondence between the symmetric Nash equilibria of $SG(C(q, M))$ which use with positive probability the last column of $C(q, M)$, and the set of solutions to $LCP(q, M)$. We follow this with a theorem that establishes a one-to-one correspondence between the symmetric Nash equilibria of $G(C(q, M))$ which are not using the last column of $C(q, M)$ and certificates for M for not belonging to $\mathbf{R}(e)$.

THEOREM 2.

(i) *Given* $\begin{pmatrix} u \\ t \end{pmatrix} \in SNE_+(C(q, M))$*, let* $z = u\frac{1}{t}$*. Then,* $z \in SOL(q, M)$*.*

(ii) *Given* $z \in SOL(q, M)$*, let* $t = \frac{1}{e^\intercal z + 1}$*;* $u = zt$*. Then,* $v = \begin{pmatrix} u \\ t \end{pmatrix} \in SNE_+(C(q, M))$*.*

Proof. Throughout the proof we denote $C(q, M)$ by C.

(i) Since $t > 0$, then, by (7d) (with $i = n$), $t = v^\intercal Cv$. Thus, by (7a)–(7b), $Mzt + (q+e)t \geq et$, $zt \geq 0$. Dividing by t, we get $Mz + q \geq 0$, $z \geq 0$. In addition, by (7d) (for $i = 1, \ldots, m$), $u_i(M_{i\bullet}u + q_i t + t - t) = 0$, so dividing by t^2, substituting for z, and summing over m, we get $0 = \sum_{i=1}^m z_i(M_{\bullet i}z + q_i) = z^\intercal(Mz + q)$.

(ii) By (1a), and setting $t = \frac{1}{e^\intercal z+1}$, $u = zt$, we have

$$\begin{pmatrix} M & q+e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} zt \\ t \end{pmatrix} \geq \begin{pmatrix} e \\ 1 \end{pmatrix} t, \quad \begin{pmatrix} zt \\ t \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and obviously $(e^\intercal z)t + t = (e^\intercal z + 1)t = 1$. In addition,

$$v^\intercal Cv = u^\intercal(Mu+qt+et)+t^2 = t^2(z^\intercal(Mz+q)+e)+1)$$

Thus, since by (1b), $z^\intercal(Mz+q) = 0$, $v^\intercal Cv = t^2(z^\intercal e + 1) = t$. Hence, $v = \begin{pmatrix} u \\ t \end{pmatrix}$ satisfies (7a)–(7c), and since $t > 0$, we have $v \in SNE_+(C(q, M))$. □

THEOREM 3.

(i) *If* $\begin{pmatrix} u \\ 0 \end{pmatrix} \in SNE_0(C(q, M))$*, then* $u \in SOL(e\tau, M)/\{0\}$ *for some* $\tau \geq 0$*.*

(ii) *Let* $y \in SOL(e\tau, M)/\{0\}$ *for some* $\tau \geq 0$*. Then, setting* $u = y\frac{1}{e^\intercal y}$*, we have* $v = \begin{pmatrix} u \\ 0 \end{pmatrix} \in SNE_0(C(q, M))$*.*

Proof. Throughout the proof we denote $C(q, M)$ by C.

(i) By (7a)–(7b), $Mu \geq e(u^\intercal Mu)$, $0 \geq u^\intercal Mu$, $u \geq 0$, and $e^\intercal u = 1$. Setting $\tau = -u^\intercal Mu$, we get

$$Mu + e\tau \geq 0, \; 0 \neq u \geq 0.$$

Moreover, since $e^\intercal u = 1$, we have (by (7d)) that $u^\intercal(Mu+e\tau) = 0$, concluding that $u \in SOL(e\tau, M)/\{0\}$ for some $\tau \geq 0$.

(ii) Noticing that $y \neq 0$ and by (1a)–(1b),

$$\begin{pmatrix} M & q+e \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ 0 \end{pmatrix} \geq \begin{pmatrix} e \\ 1 \end{pmatrix}\frac{-\tau}{e^\intercal y}, \begin{pmatrix} u \\ 0 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and $u^\intercal(Mu + e\frac{\tau}{e^\intercal \tau}) = 0$. In addition,

$v^\intercal Cv = u^\intercal Mu = \frac{-\tau}{e^\intercal y}$. Thus, $\begin{pmatrix} u \\ 0 \end{pmatrix}$ satisfy (7a)–(7b). Noticing that $e^\intercal u = 1$, completes the proof. □

Considering the definition of $\mathbf{R}(e)$, and combining Propositions 2 and 3, we get the main result of this section: that the problem $\mathbf{R}(e) - LCP(q, M)$ can be reduced to finding a symmetric Nash equilibrium for the symmetric game whose cost matrix is $C(q, M)$. Specifically, given $M \in \mathcal{R}^{m\times m}$ and $0 \not\leq q \in \mathcal{R}^m$, we search for $\begin{pmatrix} \bar{u} \\ \bar{t} \end{pmatrix} \in SNE(C(q, M))$. Note that $SNE(C(q, M) \neq \emptyset$, and since $0 \not\leq q$, $\bar{u} \neq 0$. If $\bar{t} > 0$ then $\bar{u}\frac{1}{\bar{t}} \in SOL(q, M)$, else $\bar{u} \in SOL(e\bar{\tau}, M)$, where $\tau = -\bar{u}^\intercal M\bar{u} \geq 0$, implying that $\bar{u}$ certifies that $M \notin \mathbf{R}(e)$.

Remarks

1. The Nash equilibria of the resulting symmetric game correspond to all the end points (sources or sinks) of the $\mathcal{PPAD}$ graph associated with $R(e) - LCP(q, M)$.
2. The class of all matrices M for which $LCP(q, M)$ is guaranteed to have a solution for all q is called $\mathbf{Q}$. The largest known class $\boldsymbol{Y}$ which is contained in $\mathbf{Q}$ and for which it is known that $\boldsymbol{Y} - LCP(q, M)$ is Lemke-processable, is $r(e)$.
3. Since $\mathbf{P}, \mathbf{C} \subset \mathbf{E} \subset \mathbf{R}(e)$, the reduction is applicable to $\mathbf{Y} - LCP$ where $\boldsymbol{Y}$ is $\mathbf{P}$, $\mathbf{C}$ or $\mathbf{E}$. Note that $LCP(q, M)$ has a unique solution for all q if and only if $M \in \mathbf{P}$, and that $LCP(q, M)$ has s unique solution for all $q \geq 0$ if and only if $M \in \mathbf{E}$.
4. The reduction presented in this section is a bijection between the solutions of an LCP and the Nash equilibria that uses with positive probability the pure strategy corresponding to the last column of the cost matrix of the associated symmetric bimatrix game. This bijection can expand the reduction applicability. For example, one can apply any algorithm which finds all Nash equilibrium points to find all the solutions of $LCP(q, M)$ where $M \in \mathbf{R}(e)$.

6. REDUCING LCP WITH GUARANTEED INFEASIBILITY CERTIFICATE

In this section we consider $LCP(q, M)$ for which there is no guarantee that a solution exists. To reduce such problems to bimatrix games we introduce the *augmented* problem $LCP(\tilde{q}, \tilde{M})$ associated with $LCP(q, M)$, where

$$\tilde{M} = \begin{pmatrix} M & e \\ -e^\intercal & 2 \end{pmatrix}, \tilde{q} = \begin{pmatrix} q \\ \beta \end{pmatrix},$$

and where $\beta > e^\intercal\bar{x} + \bar{s}$ for any AC vertex (not necessarily feasible) $(\bar{x}, \bar{s})$ of $F(q, M)$.

Remarks

1. It is a standard result in LP theory that if the entries in q, M are rational then β is of size polynomial in the size of q, M, and that β can be calculated in time polynomial in m.
2. Augmented LPC systems where $\tilde{M}_{m+1,m+1}$ is equal to 0 (see [CPS92]) or -1 [Tod73] are used in the LCP literature to eliminate secondary rays. Such augmentation does not work in our case since the reduction of $LCP(\tilde{q}, \tilde{M})$ to a symmetric bimatrix game would yield a pure Nash equilibrium (using with probability 1 the strategy corresponding to the last column of $\tilde{M}$) which yields no information about the solution (or lack thereof) of the original $LCP(q, M)$. To avoid this possibility, we need $\tilde{M}_{m+1,m+1} > 1$, hence the choice of 2.

In the following theorem we establish the relationship between $LCP(q, M)$ and $LCP(\tilde{q}, \tilde{M})$.

THEOREM 4. *Given $LCP(q, M)$, suppose (1) and (2) are nondegenerate. Then,*

(i) $\begin{pmatrix} \bar{z} \\ 0 \end{pmatrix} \in SOL(\tilde{q}, \tilde{M})$ *if and only if* $\bar{z} \in SOL(q, M)$.

(ii) *If* $\begin{pmatrix} \tilde{z} \\ \tilde{s} \end{pmatrix} \in SOL(\tilde{q}, \tilde{M})$ *then there exists* $(\bar{z}, \bar{s}, \bar{u}, \bar{r}) \in SR(q, M)$ *and* $\tilde{\lambda}$ *such that* $\begin{pmatrix} \tilde{z} \\ \tilde{s} \end{pmatrix} = \begin{pmatrix} \bar{z} \\ \bar{s} \end{pmatrix} + \tilde{\lambda}\begin{pmatrix} \bar{u} \\ \bar{r} \end{pmatrix}$.

(iii) *Let* $(\bar{z}, \bar{s}, \bar{u}, \bar{r}) \in SR(q, M)$. *Then there exists* $\tilde{\lambda}$ *such that* $\begin{pmatrix} \bar{z} \\ \bar{s} \end{pmatrix} + \tilde{\lambda}\begin{pmatrix} \bar{u} \\ \bar{r} \end{pmatrix} \in SOL(\tilde{q}, \tilde{M})$.

Proof.

(i) The 'only if' direction is obviously true. The 'if' direction is similarly true considering the nondegeneracy assumption (so $\tilde{z}$ is a a vertex of $F(q, M)$) and by the definition of β.

(ii) Let $\begin{pmatrix} \tilde{z} \\ \tilde{s} \end{pmatrix} \in SOL(\tilde{q}, \tilde{M})$ where $\tilde{s} > 0$. Then

$M\tilde{z} + e\tilde{s} + q \geq 0,\ \tilde{z} \geq 0,\ \tilde{s} \geq 0$, and

$\tilde{z}^\intercal(M\tilde{z} + e\tilde{s} + q) = 0,\ \tilde{s}(e^\intercal\tilde{z} - 2\tilde{s} - \beta) = 0,$

which implies that $(\tilde{z}, \tilde{s}) \in F(q, M)$ and (since $\tilde{s} > 0$) $e^\intercal\tilde{z} = \beta + 2\tilde{s}$. However, by the definition of β, $(\tilde{z}, \tilde{s})$ must be a point on a secondary ray of $F(q, M)$.

(iii) Any point on a secondary ray belongs to $F(q, M)$ which implies that for all $\lambda \geq 0$, $\begin{pmatrix} \bar{z} \\ \bar{s} \end{pmatrix} + \lambda\begin{pmatrix} \bar{u} \\ \bar{r} \end{pmatrix}$ satisfies all constraints of $LCP(\tilde{q}, \tilde{M})$ except for the last constraint. However, since $e^\intercal\bar{z} + 2\bar{s} < \beta$ and $\bar{u} \neq 0$, setting $\tilde{\lambda} = \frac{\beta - e^\intercal\bar{z} + 2\bar{s}}{e^\intercal\bar{u} + 2\bar{r}}$ yields $e^\intercal(\bar{z} + \tilde{\lambda}\bar{u}) + 2(\bar{s} + \tilde{\lambda}\bar{r}) = \beta$ which completes the proof. □

Next, we show that $M \in \mathbf{R_0}(e)$ implies that $\tilde{M} \in \mathbf{R}(e)$ which allows us to apply the reduction of the previous section to the augmented problem.

THEOREM 5. *If $M \in \mathbf{R_0}(e)$ then $\tilde{M} \in \mathbf{R}(e)$.*

Proof. Suppose $\tilde{M}(e) \notin \mathbf{R}(e)$. Then, there exists $\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} \bar{u} \\ \bar{r} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\bar{\tau} \geq 0$, such that

$$\begin{pmatrix} M & e \\ -e^\intercal & 2 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{r} \end{pmatrix} + \begin{pmatrix} e \\ 1 \end{pmatrix} \bar{\tau} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{and}$$

$$(\bar{u}^\intercal\ \bar{r}) \left[\begin{pmatrix} M & e \\ -e^\intercal & 2 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{r} \end{pmatrix} + \begin{pmatrix} e \\ 1 \end{pmatrix} \bar{\tau} \right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, $M\bar{u} + e(\bar{r} + \bar{\tau}) \geq 0,\ \bar{u}^\intercal(M\bar{u} + e(\bar{r} + \bar{\tau})) = 0$, and $\bar{r}(-e^\intercal\bar{u} + 2\bar{r} + \bar{\tau}) = 0$. Moreover, since from the last inequality above, $-e^\intercal\bar{u} + 2\bar{r} + \bar{\tau} \geq 0$ we have $\bar{r} + \bar{\tau} > 0$ (as otherwise $\bar{u} = 0, \bar{r} = 0$). Thus $\bar{u} \in SOL((\bar{r} + \bar{\tau})e, M) \setminus \{0\}$, so (since $\bar{r} + \bar{\tau} > 0$) $M \notin \mathbf{R_0}(e)$. □

Combining Propositions 4 and 5, and recalling the definition of the class **USR**, we get that we can reduce $\mathbf{USR} - LCP(q, M)$ to the problem of finding a symmetric Nash equilibrium for the symmetric game with cost matrix

$$C(\tilde{q}, \tilde{M}) = \begin{pmatrix} M & e & q+e \\ -e^\intercal & 2 & -\beta+1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, let $\begin{pmatrix} \bar{z} \\ \bar{s} \\ \bar{t} \end{pmatrix} \in SNE(C(\tilde{q}, \tilde{M}))$. We then conclude that:

1 $\bar{\boldsymbol{t}} = \mathbf{0}$. Then by Proposition 3, $\tilde{M} \notin \mathbf{R}(e) \Rightarrow$ (by Proposition 5) $M \notin \mathbf{R_0}(e) \Rightarrow M \notin \mathbf{USR}(e)$.

2 $\bar{\boldsymbol{t}} > \mathbf{0}$. Then (by Proposition 2) $\begin{pmatrix} \bar{z}\frac{1}{\bar{t}} \\ \bar{s}\frac{1}{\bar{t}} \end{pmatrix} \in SOL(\tilde{q}, \tilde{M})$.

- **2.1** $\bar{\boldsymbol{s}} = \mathbf{0}$. Then (by Proposition 5), $\bar{z}\frac{1}{\bar{t}} \in SOL(q, M)$.
- **2.2** $\bar{\boldsymbol{s}} > \mathbf{0}$. Then (by Proposition 5), we have $(\bar{z}, \bar{s}, \bar{u}, \bar{r}) \in SR(q, M)$.
  - **2.2.1** $\bar{\boldsymbol{r}} > \mathbf{0}$. Then, $M \notin \mathbf{USR}$.
  - **2.2.2** $\bar{\boldsymbol{r}} = \mathbf{0}$. Check whether $FEAS(q, M) = \emptyset$. That is, find either
    (a) $y \geq 0$ such that $y^\intercal M \leq 0,\ y^\intercal q < 0$; or
    (b) $z \geq 0$ such that $Mz + q \geq 0$.[4]
    In case (a), by Farkas' Lemma (see [CPS92], 2.7.9), $FEAS(q, M) = \emptyset \Rightarrow SOL(q, M) = \emptyset$. In case (b) (where $FEAS(q, M) \neq \emptyset$) we conclude that $M \notin \mathbf{USR}$, as the coexistence of $(\bar{z}, \bar{s}, \bar{u}, 0) \in SR(q, M)$ with $M \in \mathbf{USR}$ implies that $FEAS(q, M) = \emptyset$.

[4]Note that, as it is well known, this check can be carried out in time polynomial in the size of the given problem.

Figure 1 depicts several well studied classes of matrices M (in addition to those discussed in the remarks of the preceding section) for which $LCP(q, M)$ can be reduced, as described in this section, to a symmetric bimatrix game. Several of the classes have interesting characteristics as follows:

- See the second remark at the end of the previous section regarding the matrix classes **P** and **E**.
- $M \in \mathbf{CSU}$ (column sufficient) if and only if $SOL(q, M)$ is convex for all q (see [CPS92], 3.5.8).
- $M \in \mathbf{RSU}$ (row sufficient) if and only for all q, every Karush-Kuhn-Tucker point of the quadratic optimization problem associated with $LCP(q, M)$ is also a solution for $LCP(q, M)$ (see [CPS92], 3.5.4).
- The convex quadratic programming problem is an optimization problem with convex quadratic objective function and linear constraints. Since the convex quadratic programming problem can be formulated as an $LCP(q, M)$ where $M \in \mathbf{PSD}$ (the class of positive-semidefinite matrices), we conclude that any convex quadratic programming problem can be reduced to a symmetric 2-NASH.

7. CONCLUDING REMARKS

1. The results of Section 5 can be extended to $\mathbf{R}(d)$ for any $d \in \boldsymbol{\mathcal{R}}^{\boldsymbol{m}}_{++}$. The key to the extension is the following proposition which can be readily verified by considering the definitions of $\mathbf{R}(e)$ and $SOL(q, M)$.

   PROPOSITION 6. *Let $d \in \boldsymbol{\mathcal{R}}^{\boldsymbol{m}}_{++}$, and let D be a diagonal matrix whose diagonal entries are the entries of d.*

   (i) *$M \in \mathbf{R}(d)$ if and only if $D^{-1}MD \in \mathbf{R}(e)$.*
   (ii) *$z \in SOL(q, M)$ if and only if $Dz \in SOL(D^{-1}q, D^{-1}MD)$.*

   Armed with the preceding theorem and replacing $F(q, M)$ with

   $F(d, q, M) \triangleq \{z \in \boldsymbol{\mathcal{R}}^{\boldsymbol{m}}_{+},\ s \in \boldsymbol{\mathcal{R}}_{+} |\ Mz + ds + q \geq 0,\ z \geq 0\}$,

   and by following the arguments in Section 5, we get that given $d \in \boldsymbol{\mathcal{R}}^{\boldsymbol{m}}_{++}$ we can reduce $\mathbf{R}(d) - LCP$ to a symmetric bimatrix game with cost matrix

   $$\begin{pmatrix} D^{-1}MD & D^{-1}q + e \\ 0 & 1 \end{pmatrix}.$$

   Similarly (but with more detailed constructions which we'll not present here), it is possible to extend the reduction in Section 6 to $\mathbf{USR}(d) - LCP$, where $\mathbf{USR}(d)$ is defined as the set of matrices $M \in \mathbf{R_0}(d)$ for which the existence of a secondary ray of $F(d, q, M)$ with $r = 0$ implies that $FEAS(q, M) = \emptyset$.

2. The reductions in Sections 5 and 6 are simple and easy to execute. Thus, any algorithm that is applicable to bimatrix games can be directly used to solve $\boldsymbol{Y} - LCP(q, M)$ where $\boldsymbol{Y} = \mathbf{R}(e)$ (as described in Section 5) and where $\boldsymbol{Y} = \mathbf{USR}$ (as described in Section 6). In addition, because the reductions are bijections,

bimatrix game algorithms with different goals, such as enumerating all, or specific subsets of the solutions, can be used for similar goals for the linear complementarity problems for which our reductions are applicable.

3. Over the years several refinements of Nash equilibrium have been introduced. In particular, some results regarding the existence and computation of these refinements have been established. In [MT98] some of these refinements are generalized to LCPs. Given the reduction of $LCP(q, M)$ with $M \in \mathbf{USR}$ to the problem of finding Nash equilibrium it provides us with a tool to investigate analogous questions with respect to the generalized refinements to $LCP(q, M)$ with $M \in \mathbf{USR}$. In the full version of this extended abstract, we demonstrate such an analysis by proving that any $LCP(q, M)$ with $M \in \mathbf{R_0}(e)$ has a proper solution. As a corollary of this analysis we proved that the (unique) solution of $LCP(q, M)$ where $M \in \mathbf{P}$, is proper and thus setteled a conjecture posed in [MT98].

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APPENDIX

A. CLASSES OF MATRICES

We use [CPS92] as the primary reference for these definitions. Suppose $M \in \boldsymbol{\mathcal{R}}^{n \times n}$.

M is a *positive semi-definite* (**PSD**) matrix if $x^T M x \geq 0$ for any x.

M is a *positive definite* (**PD**) matrix if $x^\intercal M x > 0$ for any $x \neq 0$.

M is a $\mathbf{P_0}$ matrix if all its principal minors are nonnegative.

M is a **P** matrix if all its principal minors are positive.

M is a *column-sufficient* (**CSU**) matrix if for any vector x and for every $i = 1, 2, \ldots .m$, $x_i(Mx)_i \leq 0$, then $x_i(Mx)_i = 0$ for all $i = 1, 2, \ldots, m$.

M is a *row-sufficient* (**RSU**) matrix if its transpose is a **CSU** matrix.

M is a *sufficient* (**SU**) matrix if $M \in \mathbf{RSU} \cap \mathbf{CSU}$.

M is a *copositive* ($\mathbf{C_0}$) matrix if $x^\intercal M x \geq 0$ for any $x \geq 0$.

M is a *strictly copositive* (**C**) matrix if $x^\intercal M x > 0$ for any $0 \neq x \geq 0$.

M is a *copositive-plus* ($\mathbf{C_0^+}$) matrix if M is copositive and $x^\intercal M x = 0,\ x \geq 0$ implies that $(M + M^\intercal)x = 0$.

M is a *copositive-star* ($\mathbf{C_0^*}$) matrix if M is copositive and $x^\intercal M x = 0,\ Mx \geq 0,\ x \geq 0$ implies that $M^\intercal x \leq 0$.

M is a *semimonotone* ($\mathbf{E_0}$) matrix if for any non-zero $x \geq 0$, there exists an index k such that $x_k > 0$ and $(Mx)_k \geq 0$.

M is a *strictly semimonotone* (**E**) matrix if for any non-zero $x \geq 0$, there exists index k such that $x_k > 0$ and $(Mx)_k > 0$.

M is a $\mathbf{E_1}$ matrix if for every nonzero vector $z \in SOL(0, M)$, there exists non-negative diagonal matrices D_1 and D_2 such that $D_2 z \neq 0$ and $(D_1 M + M^T D_2)z = 0$.

M is a **L** matrix if M is $M \in \mathbf{E_0} \cap \mathbf{E}1$.

M is a $\mathbf{Q_0}$ matrix if $SOL(q, M) \neq \emptyset$ for all q for which $FEAS(q, M) \neq \emptyset$.