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On the Existence of Solutions in Modular Fixturing *

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Abstract

A fixture is a device that locates and holds parts during machining or assembly. A *modular* fixture employs reusable components on a regular lattice. Given a part, machinists combine intuition with trial-and-error to design an appropriate fixture. When a machinist is unable to find a design, it may be the case that (1) a feasible design was overlooked or (2) no feasible design exists. Complete algorithms for modular fixturing, such as [Brost and Goldberg 1994], insure that no fixture design is overlooked. But the question remains: are there parts for which no modular fixture exists?

For the class of modular fixtures using 3 locators and a clamp, we show that there exists a class of polygonal parts that cannot be fixtured. We believe that this is the first negative result in the area of fixturing. We also show two positive results, namely that a modular fixture always exists when we broaden the class of fixtures to include T-slot and narrow the class of parts. We show that one class of fixtures strictly dominates the other. These results raise a number of open problems concerning the existence of solutions for other classes of fixtures and parts and suggest a hierarchy of fixturing models.

1 Introduction

A fixture is a device that locates and holds a part for machining, assembly, inspection, etc. *Modular* fixture systems typically offer a precision lattice of holes for receiving locators and clamps, thereby insuring precise location and allowing re-use of the fixture for a variety of parts.

> *"...This type of tooling may revolutionize the way our manufacturing industries produce parts. The overall versatility, adaptability and compatibility of these systems with new manufacturing methods will dramatically improve productivity by increasing production rates and substantially reducing the lead time required to fabricate virtually any part. " [Hoffman 1987]*

Before adopting a new class of fixtures, it is natural to want clarification on the class of parts for which the new fixtures will be suitable. This paper explores this question in the context of 2 classes of modular fixtures. Although these classes include a small subset of the components found in a commercial modular fixturing system, the results suggest a number of open problems for other classes of fixtures and point the way toward a hierarchy of fixture classes.

For a given operation, one face of a part is generally specified to be in contact with a planar support surface. In this paper we consider the projection of the part into this plane, and treat only the planar problem. In particular, we assume that the input is a polygonal projection of some part and refer to this projection as the *part*.

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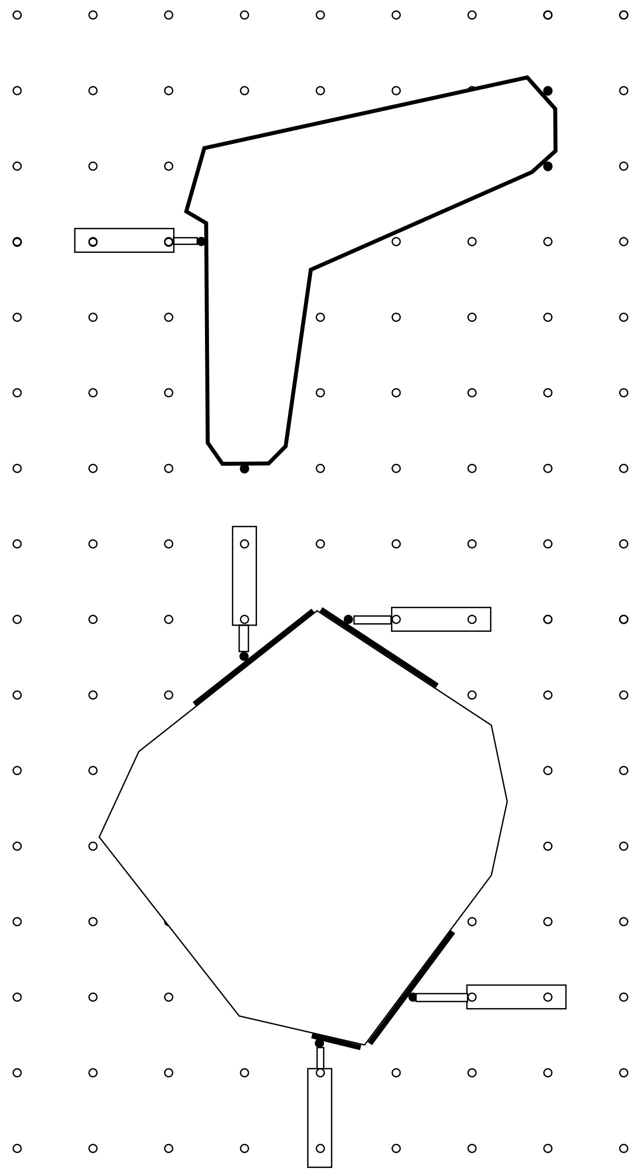

Figure 1: *Examples of parts in modular fixtures. At top, a part fixtured with 3 locators and one clamp. At bottom, a part fixtured using 4 clamps.*

Given a class of fixtures and a part, does a fixture exist that will hold this part in form closure? If so, we might say that the part is *fixturable*. Examples of fixturable parts are illustrated in Figure 1. Most literature on fixturing assumes that all parts are fixturable. In experiments with an algorithm for finding fixtures, we found that for most polygonal parts, many fixtures exist; random experiments did not yield a single part that was not fixturable. We thus set out to prove that all polygonal parts are fixturable. Instead, we discovered a class of counterexamples.

In this paper we consider two classes of modular fixtures. Both prevent a part from translating or rotating in the plane using four point contacts on the part's boundary. We consider a planar polygonal slice of the 3D part and refer to this polygon as the part. An arrangement of four contacts on a regular lattice constitutes a *fixture*. We assume that all contacts are unilateral point constraints and that all contacts are frictionless. This is conservative, since any fixture that achieves form closure without friction will also achieve form closure with non-zero friction. We also assume that all contacts are interior to an edge of the part – we do not allow a fixture element to contact a part vertex since such contacts may damage the part. A fixture is acceptable if it provides *form closure*, which is a kinematic condition that prevents any infinitesimal motion [Reuleaux 1876]. Results from linear algebra show that at least 4 wrenches are necessary for form closure. [Markenscoff, Ni, and Papadimitriou 1990] showed that 4 wrenches are sufficient for any piecewise-smooth compact connected planar body, excluding surfaces of revolution.

We consider two types of modular contacting elements: *locators* are cylinders rigidly attached to the baseplate through pegs that fit into a square lattice of holes. *Clamps* are adjustable elements that can make contact at any point within a unit distance horizontally or vertically from a lattice hole. These are formalized in the next section. A clamp is geometrically equivalent to the modular elements commonly known as "T-slots" in the field [Hoffman 1987]. Note that clamps can be more easily adapted than locators to reach part boundaries. For the same reason, clamps are less capable of accurately locating parts relative to the baseplate.

A *fixel* (fixture element) refers to either a locator or a clamp. We use the term *3L/1C fixture* to refer to an arrangement of 3 locators and 1 clamp. Similarly, we use *0L/4C* to refer to an arrangement of 4 clamps. Section 4 describes our primary result: an infinite class of polygonal parts for which no 3L/1C fixture exists. In Section 5, we prove two positive results: infinite classes of polygonal parts for which a 0L/4C fixture is guaranteed to exist.

Although fixels have finite radius, we initially assume that they are points. In Section 6, we generalize our results to fixels of non-zero radius.

2 Related Work

The problem of designing modular fixtures is closely related to grasp planning in robotics. The goals of both are similar: fixing an object kinematically by means of a suitable set of contacts. The primary difference is that modular fixtures place restrictions on the relative location of contacts due to the underlying lattice.

The century-old definition of *form closure* [Reuleaux 1876] captures the intuitive function of a fixture. A set of contacts provides form closure if infinitesimal part motion is completely constrained; equivalently, the set of contacts is able to resist arbitrary forces and torques on the part. Each contact provides a *wrench*: a force with a point of application. In the plane, a wrench can be represented as a vector in $\mathbb{R}^3$, where the first two components represent the direction of force and the third component represents a moment about an arbitrary origin [Ohwovoriole 1987]. A set of wrenches provides form closure if it positively spans $\mathbb{R}^3$.

For 4 co-planar contacts, Nguyen [Nguyen 1988] gave a geometric test for form-closure[1] and gave the formal definition of form closure repeated in the next section. Nguyen also showed how to construct grasps with four frictionless fingers by finding 4 independent regions on the boundary of the part such that if each region contains one contact, the part will be in form closure. For three frictional contacts in the plane, Ponce and Faverjon [Ponce 1993] showed that comparable regions on a polygon could be found using linear optimization. Asada and By [Asada and By 1985] showed how to determine whether a given fixture design provides total constraint of a rigid body, as well as loading accessibility before clamping. Surveys of the grasping literature can be found in [Pertin-Troccaz 1989, Grupen, Henderson, and McCammon 1989].

Several algorithms on synthesizing modular fixtures have been developed recently. Brost and Goldberg [Brost and Goldberg 1994] presented a design algorithm that is complete in the sense that it is guaranteed to find a *3L/1C* solution for a given polygon if one exists, and to return failure otherwise. That algorithm runs in time $O(n^5 d^5)$, where n is the number of part edges and d is its diameter in lattice units.

Penev and Requicha [Penev and Aristides 1995] studied the related problem of *fixture foolproofing*: given a 3L/1C fixture, where should blocking pins be inserted to insure that the part can be loaded in only one desired pose? Wallack and Canny

[1] Nguyen used the term "force-torque closure" to describe what is more commonly called form closure [Trinkle 1992].

[Wallack and Canny 1994] presented a complete algorithm for a fixturing model using four locators on a split lattice that can be closed like a vise.

Overmars *et al.* [Overmars *et al.*] recently proposed a new class of planar fixtures that includes flat edge-contacts. They presented a complete algorithm for finding such fixtures that runs in time $O(n(n+p)^{4/3+\epsilon} + K)$, where K is the number of fixtures found, and p is the part's perimeter in lattice units. As we do in this paper, they considered the class of parts for which such fixtures exists and showed that any polygonal part that has no edge parallel to one of the edges of its convex hull can be fixtured using one edge and two point contacts.

Mishra [Mishra, Schwartz, and Sharir 1987] applied work on grasping to the problem of designing modular fixtures using a regular lattice. Mishra gave the first existence results [Mishra 1991] as described in Section 5.1 below.

This paper is a substantially revised and extended version of [Zhuang, Goldberg, and Wong 1994].

3 Preliminaries

We define a *regular lattice* as a 2D plane on which the lattice sites are arranged in rows and columns with uniform unit distance. Henceforth, all distances are given in units of lattice spacing. We assume that the lattice is larger than the part to be fixtured. Since contacts are frictionless, they can only exert force normal to the contacted edge of the part.

As stated earlier we focus on planar cross-sections of 3D parts. By a *part* we mean a regular [Requicha 1980] planar set with a boundary that is connected and piecewise smooth. We denote the boundary of a part P by ∂P.

We denote the set of all lattice sites as integer pairs, $\mathbf{Z}^2$. A triplet of lattice sites is *non-trivial* if its components do not lie on a straight line. A pair of lattice sites has length equal to the distance between them. If two pairs of lattice sites define the line segments that are parallel and of equal length, we say that the two pairs are equivalent. The distance between a point x and a set S is defined as $d_S(x) = \inf\{d(x,y) \mid y \in S\}$, where d is the Euclidean distance given in units of lattice spacing.

Given a part P whose 2D boundary is ∂P, we want to find a set of contacts $\mathcal{F} \subset \partial P$ such that P will be held in form closure, *i.e.* ∂P is smooth at each contact and the contact normals, n_1, n_2, n_3 and n_4, satisfy the following [Nguyen 1988]

1. any 3 of them do not intersect at a common point or at infinity,
2. let p_{12} and p_{34} be the intersection points of n_1 and n_2, and of n_3 and n_4 respectively, then

$$p_{34} - p_{12} = \pm(\alpha n_1 + \beta n_2) = \mp(\gamma n_3 + \delta n_4)$$

   for some $\alpha, \beta, \gamma, \delta > 0$. Namely the intersection of the first two normal vectors should be covered by the cone made of the other two normal vectors, and vice versa.

Contacts are realized using fixels: either locators or clamps. Recall that locators are rigid cylinders attached to lattice sites and so all locator contacts must be $\in \mathbf{Z}^2$. Clamps must be anchored at a lattice site but can be extended along the principle axes within a unit. For each clamp contact, u, $\exists$ (distinct) $v \in \mathbf{Z}^2$ such that:

- $\overline{uv}$ is parallel to one of the principle axes.
- $\overline{uv} \cap P = \emptyset$,
- $d(u,v) \leq 1$.

4 Unfixturable Parts

3L/1C fixtures use three contacts at locators (ie lattice sites). Obviously, a part small enough to fit between 4 adjacent lattice sites cannot make contact with more than one locator; for such parts a 3L/1C fixture does not exist. Similarly, it is easy to construct a long but very thin part that is not fixturable. The question is: can we construct a part of arbitrary "size" that is unfixturable? First, we formalize the notion of size in terms of part width.

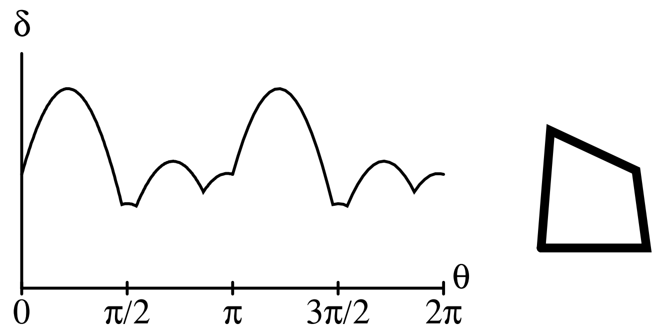

Figure 2: *The width of a part at direction θ is the maximum distance between two parallel supporting lines (indicated by δ in the figure and w elsewhere in the paper).*

Classical computational geometry defines a *width function* [Yaglom and Boltyanskii 1961, Benson 1966, Lyusternik 1966] for a planar part, S, as the distance between two parallel supporting lines (See Figure 2). The maximum and minimum values of this function are well defined; we denote them with $\overline{w}(S)$ and $\underline{w}(S)$, respectively. To characterize parts that cannot fall through cracks in the lattice, we define the *size* of a part as its minimum width.

We now construct a polygonal part of arbitrary size that is unfixturable. For any given size M, we first construct a disk of size $> M$ that can make contact with at most 2 lattice sites. We then show how to transform this disk into a regular polygon that preserves this property. Thus this polygon cannot be fixtured under the 3L/1C model.

Three noncolinear points uniquely determine a circle. Thus, every non-trivial triplet of lattice sites, t, determines a disk which we denote by $d(t)$. If two triplets determine a disk of the same width, we say that these triplets are *equivalent*. Let the *maximum width* of a triplet, $\overline{w}(t)$, be the length of the longest side of the triangle it determines.

Lemma 1 *For any given width there exists a* disk *of greater width that can achieve contact with at most two lattice sites.*

Proof. For any given positive number M, let $D(M)$ be the set of disks with width between M and $M+1$, determined by triplets of lattice sites. $D(M)$ is finite, since the number of triplets with maximum width less than $M+1$ is finite. Let us define the set of widths as

$$W(M) = \{w(d(t)) \mid d(t) \in D(M)\}$$

Then $W(M)$ is a finite set. Let w^* be any width in the interval $(M, M+1]$ not in $W(M)$ and let D^* be the disk with width w^*. D^* is not in $D(M)$ thus can achieve contact with at most two lattice sites. *(See Figure 3)* □

Based on this disk, we now construct a polygon that can achieve contact with at most two locators.

Theorem 1 *For any given width there exists a* convex polygon *of greater width that can achieve contact with at most two lattice sites.*

Proof. By lemma 1, for any given positive number M we can construct a disk D^* with a width $w^* > M$, that has at most two lattice sites on its boundary. For any pair of lattice sites with a length less than or equal to w^*, we can always locate the disk D^* such that the two lattice sites are on its boundary. Let Q be the set of all non-equivalent pairs of lattice sites with length less than or equal to w^*. Q is a finite set, thus we can represent Q as $Q = \{p_i \mid 1 \leq i \leq k\}$, for some $k > 0$. We define $\epsilon(p)$ as the minimum distance from any non-contacting lattice sites to the boundary of the disk ∂D^*. Then we have $\epsilon(p_i) = \inf\{d_{\partial D^*}(f) \mid f \in \mathbf{Z}^2, f \notin p_i\} > 0$, for all $1 \leq i \leq k$, by the construction of the disk D^*. Let $\epsilon = \min\{\epsilon(p_i) \mid 1 \leq i \leq k\}$, then $\epsilon > 0$. We now construct a equilateral convex polygon. There exists an inscribed equilateral polygon, P, of D^*, such that the length of its side is less than $\frac{1}{2}\epsilon$. *(Figure 4)* To construct one, we must choose the number of sides of P, denoted by N, large enough. Such an N exists, since the length of the side of P, denoted by $L(N)$, satisfies the following

$$L(N) = w^* \sin\frac{\pi}{N} \to 0, \quad as \quad N \to \infty.$$

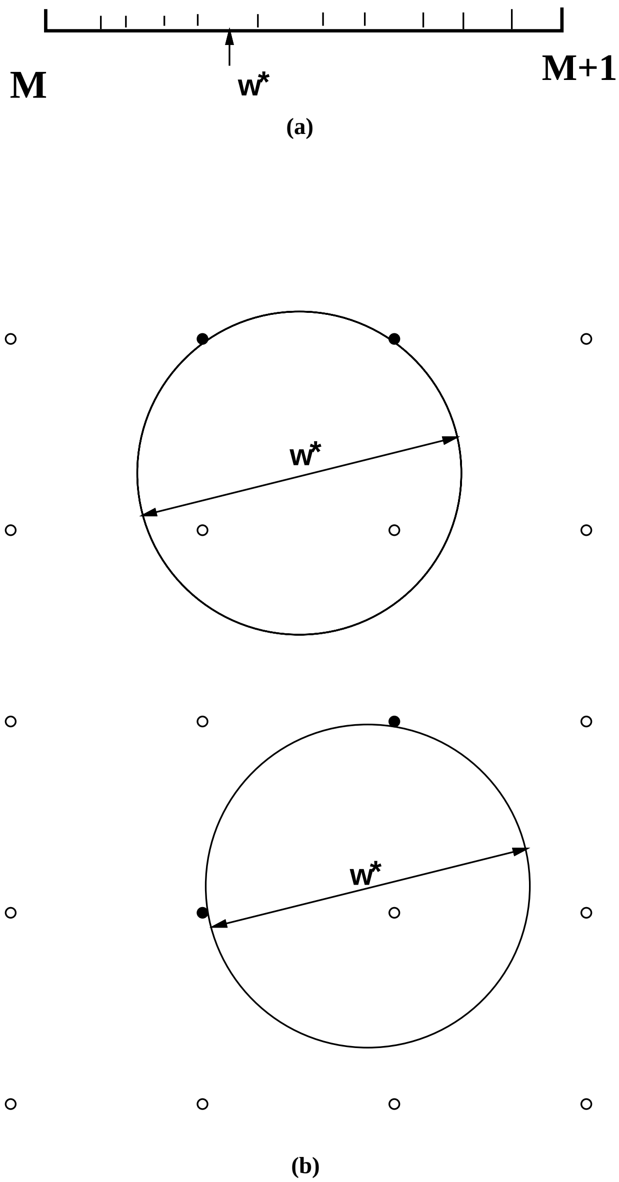

Figure 3: *Proving lemma 1: (a) The set $W(M)$ is a finite set of discrete points within the interval $(M, M+1]$, which allows us to choose an excluded width, w^*, as shown. (b) A disk of this width can achieve contact with at most two lattice sites.*

Therefore we can construct such a equilateral polygon, P, with the length of its side less than $\frac{1}{2}\epsilon$ by choosing

$$N > \frac{\pi}{\sin^{-1}\frac{\epsilon}{2w^*}}.$$

Since $w^* > M$ (by the construction of D^* in the lemma 1), we can select N to be large enough, such that $\underline{w}(P) > M$.

We denote the maximum distance between P and D^* by δ. Since $N \geq 3$, we have

$$\delta = \frac{1}{2}L(N)\tan\frac{\pi}{2N} < \frac{1}{2}L(N) < L(N) < \frac{1}{2}\epsilon.$$

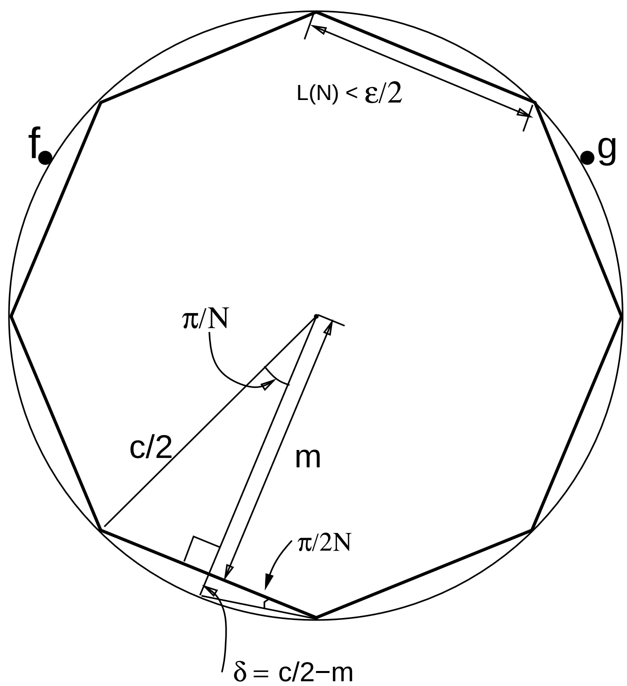

Figure 4: Illustration of how to transform the disk into an unfixturable polygon: all variables are as in the proof of the Theorem 1.

Claim: Such a polygon, P, can achieve contact with at most two lattice sites.

Proof of Claim: Let f, g and h be any three lattice sites, among which each pair has a length no greater than w^*. (If some pair has a length greater than w^*, then they cannot form a three-point contact, due to the fact that the maximum width of P is w^*.) Without loss of generality, we assume that P has f and g as its $two-point$ contact, and the contacting edges are e_f and e_g, respectively. Note that P is an inscribed polygon of D^*. We can position the disk D^*, with P inscribed, on the lattice, such that f and g are on the arcs corresponding to e_f and e_g, respectively. (This can be always achieved.) By the construction of D^*, we know that $d_{\partial D^*}(h) \geq \epsilon$. Therefore, by the construction of the equilateral polygon P, the distance from h to the boundary of P is greater than $\frac{1}{2}\epsilon$, namely $d_{\partial P}(h) > \frac{1}{2}\epsilon$. In order to achieve $two-point$ contact on the edges e_f and e_g, we can always rotate P inside the disk D^* first, then translate it to achieve the contact. After the rotation, f and g will still lie on the corresponding arcs. Therefore the distance needed for translation is less than $\frac{1}{2}\epsilon$, because the distance of the translation is bounded by the length of the side of P, which is less than $\frac{1}{2}\epsilon$. But this translation will not be enough to make h the third point contact, since h is distant from the edge of P more than $\frac{1}{2}\epsilon$. Hence P can achieve contact with at most two lattice sites. The intuition behind the above proof is illustrated in *Figure 5*. □

□

Thus we can construct an infinite set of convex polygonal parts of arbitrary size that cannot be fixtured with 3 locators and 1 clamp. We note that the value of N given above is not a lower bound on the number of sides needed to generate an unfixturable part; often fewer sides are sufficient.

5 The 0L/4C Model

Since the 3L/1C fixture model cannot fixture all polygonal parts, we consider how we might increase the range of fixtures and decrease the range of parts, such that a fixture design is guaranteed to exist. In this section, we present several positive

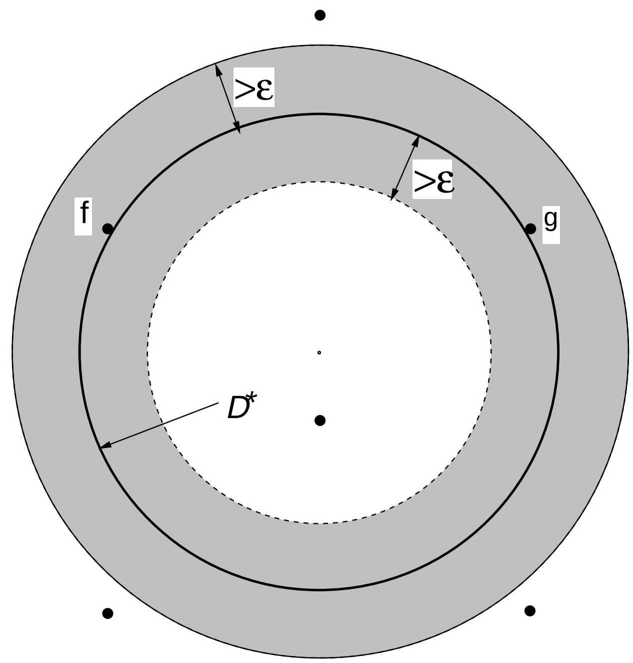

Figure 5: *Intuition behind the proof: All lattice sites except f and g are outside an annulus of width 2ϵ, where D^* is the disk constructed by lemma 1. Thus the polygon cannot be shifted to make contact with a third lattice site without losing contact with f and g.*

results.

In the 0L/4C model, we assume the same rectangular lattice as in the 3L/1C model, but instead of using 3 locators restricted to lattice sites, we allow 4 clamps. Recall that a clamp contact can translate either vertically or horizontally within one unit from a lattice site. Thus any point on the principal axes of the baseplate is available for clamping. See *Figure 1*.

Perhaps more familiar to machinists is the so-called "T-slot" modular fixturing system [Hoffman 1987] as shown in *Figure 6*. These use grooves or tracks cut into the baseplate that allow fixels to slide horizontally and vertically. We note that the T-slot system is geometrically equivalent to the 0L/4C model; all results for the latter also hold for the former.

We now consider two restricted classes of polygonal parts.

5.1 Rectilinear Parts

By a *rectilinear* part, we mean that all edges are either parallel or perpendicular to each other. Mishra showed the following result and acknowledged that improvements were likely.

Theorem [Mishra 1991] If P is a rectilinear part with all edges of length $\geq$ 4 units, then there always exists a fixture using at most 6 clamps.

In this section we tighten Mishra's result as follows:

Theorem 2 *Let P be a rectilinear part and M be the minimal enclosing rectangle aligned with the edge of ∂P. If each side of M has a length > 1, then there always exists a 0L/4C fixture for P.*

Proof. : We show that a fixture always exists for this part when it is aligned with the principal axes of the lattice.

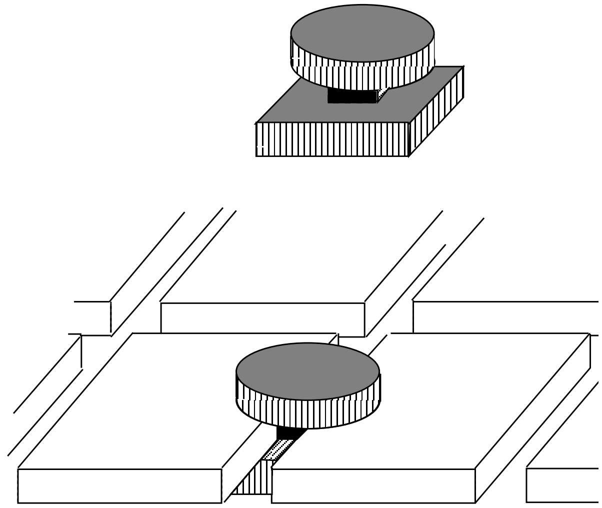

Figure 6: T-slot fixels are equivalent to clamps in our model.

We embed a unit-length square S into M also aligned with the edges of ∂P. Extend each of the four sides of S until both of its endpoints reach M. These extended lines intersect ∂P at points $b_1, b_2, \ldots, b_8$, as shown in *Figure 7*. Note that by our assumptions we can always embed S such that the points b_i are not vertices of ∂P.

We claim that the following two sets of points yield a fixture for the part: $\{b_1, b_4, b_5, b_8\}$ and $\{b_2, b_3.b_6.b_7\}$. Clearly b_1b_6 and b_2b_5 can be aligned with 2 vertical lattice lines whose separation is exactly 1 unit. A similar case holds for the lines b_3b_8 and b_4b_7. Hence, with zero rotation, it is always possible to shift ∂P horizontally and vertically such that the four points defining the form closure are each vertically or horizontally close enough to a lattice site, at which we could anchor a fixel applying a force normal to ∂P at that point. □

Note that 4 contacts are necessary according to [Reuleaux 1876].

Corollary 1 *Let P be a rectilinear part with all edges of length > 1 unit. Then there always exists a 0L/4C fixture for P.*

5.2 Convex Polygonal Parts

Nguyen [Nguyen 1988] studied the problem of achieving form closure for polygonal parts when there is no restriction on placement of contacts. Given 4 edges of the part, he showed how to find a set of 4 segments [2] such that if frictionless contacts are placed at any point along each segment, the part will be in form closure. We refer to such a set of 4 edge segments as *Nguyen set*. We note that given 4 edges, the Nguyen set is not unique: the length of one segment in the set can be traded off against the length of another segment. If we sort the segments in the set by length, $l_0 \leq l_1 \leq l_2 \leq l_3$, let l_1 be the *critical length* of the Nguyen set.

Theorem 3 *If P is a convex polygonal part that has a Nguyen set with critical length $> \sqrt{2}$, then there always exists a 0L/4C fixture for P.*

Proof. Without loss of generality we can always locate P such that its shortest Nguyen segment, l_0, is reachable by a clamp. By the conditions of the theorem, the remaining segments must each be longer than $\sqrt{2}$. By projecting each segment onto the horizontal or vertical axis, we see that we can always achieve clamp contacts along the principal axes within unit distance of a lattice site. Since P is convex, no clamps interfere and hence P has a 0L/4C fixture. □

[2] Nguyen used the term "independent regions of contact".

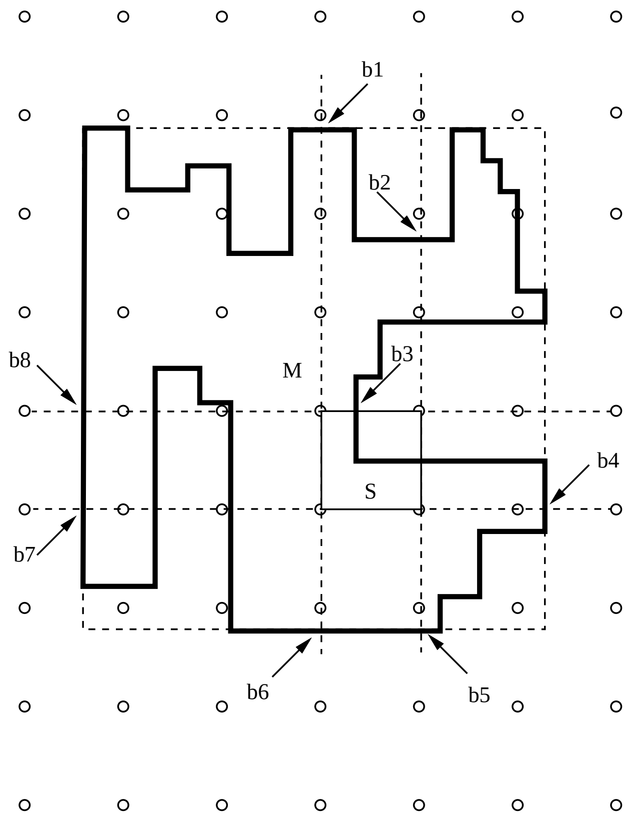

Figure 7: A rectilinear part and possible clamp positions: all variables correspond to the proof of Theorem 2.

Moreover, it follows that such parts can be fixtured in an arbitrary orientation. This would be useful, for example, to a machinist who must fixture two parts in alignment.

6 Generalization to Round Fixels

Although physical fixels are round, we assumed in previous sections that they are points. We can justify this assumption by "growing" the part by the nonzero radius of fixels using a Minkowski sum operation, while simultaneously shrinking the fixels to points (see Figure 8). Then we treat the expanded part boundary as a new part. Although vertices of the original part will form circular segments of the expanded part, we can ignore these segments since fixel-vertex contacts are not permitted under our assumptions. We must further assume that fixel radius is the same for all fixels and that this radius is less than one half of the distance between lattice sites; otherwise two adjacent fixels may intersect. In this section we generalize all results developed in the previous sections to round fixels, based on the Minkowski operation.

6.1 Unfixturable Parts on 3L/1C with Round Fixels

We can generalize Theorem 1 to round fixels by a "reverse Minkowski operation".

Let P be the convex polygonal part constructed in Theorem 1 as a counterexample for 3L/1C. We replace each vertex of P by a round corner with radius of the round fixel. Let us denote the new part by $\tilde{P}$. Note that edges of P are shortened but no straight edge will be eliminated, since fixel radius is relatively small compared to the width of P.

Thus we can consider $\tilde{P}$ as an expanded version of P^* by the Minkowski operation. Or equivalently we shrink $\tilde{P}$ to a polygon P^*, by the "reverse Minkowski operation" using the fixel radius. The set of straight edges of $\tilde{P}$ is a subset of the boundary of P, which does not have a 3L/1C fixture with point fixels. Therefore $\tilde{P}$ does not have a 3L/1C fixture with point fixels. Hence P^* is a polygon, which does not have a 3L/1C fixture with round fixels. Furthermore we can construct such a polygon larger than any arbitrarily given size, according to Lemma 1 and Theorem 1.

6.2 Rectilinear Parts with Round Fixels

Now we generalize corollary 1 to round fixels. A simple "reverse Minkowski operation" does not work for the generalization. So we present a new proof with round fixels, by using the Minkowski operation directly.

Theorem 4 *Let P be a rectilinear part with all edges of length > 1 unit. Then there always exists a 0L/4C fixture with* round fixels

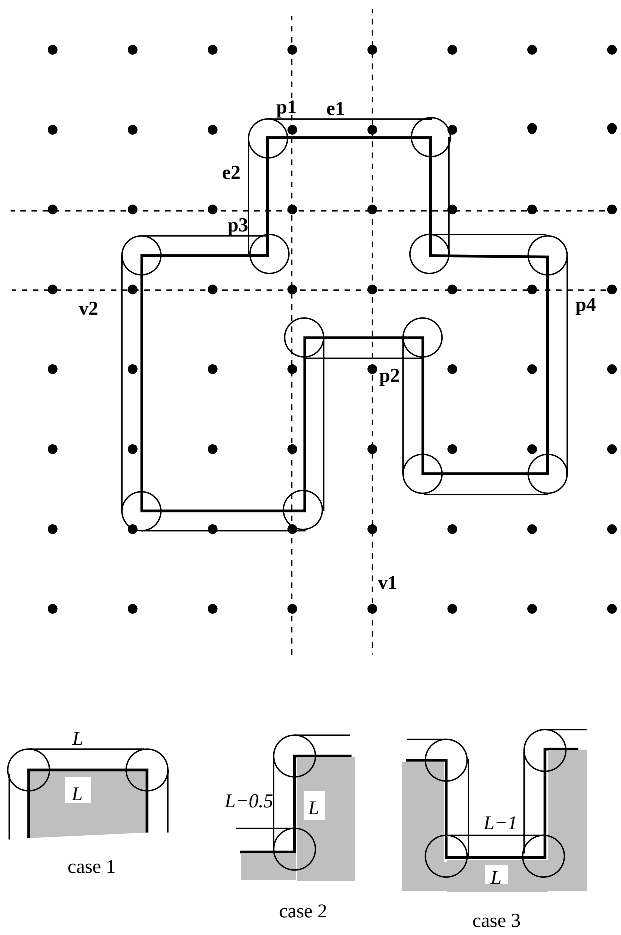

Figure 8: *The expanded rectilinear part. At the top, a grown part that can be fixtured by point fixels. At the bottom, three cases of the straight edges in the grown rectilinear part, where the shadow stands for the interior of a part. All variables are corresponding to the proof of Theorem 4.*

Proof. Let $\tilde{P}$ be the expanded part of the original part P, by the fixel radius using the Minkowski operation. Since each side of P is longer than 1 and the radius of the round fixel is less than $\frac{1}{2}$, the expanded part $\tilde{P}$ does not lose any straight edges although some of them shorten. There are three cases for the edges of P, which are illustrated in ***Figure 8***. Let L denote the length of an edge in P and $\tilde{L}$ the length of the corresponding straight edge in $\tilde{P}$. Then in the case (1), $\tilde{L} = L$; in case (2), $\tilde{L} = L - \frac{1}{2} > \frac{1}{2}$; in case (3), which is the worst case, $\tilde{L} = L - 1 > 0$. To fixture P with round fixels, we only need to fixture $\tilde{P}$ with point fixels. We will show constructively that a 0L/4C fixture with point fixels always exists for $\tilde{P}$ when it is aligned with the principal axes of the lattice. Let e_1 be the upper most horizontal edge of $\tilde{P}$. Obviously e_1 belongs to case (1). Therefore e_1 has a length $= \frac{1}{2} + \epsilon$ with $\epsilon > 0$. Without loss of generality, we choose point p_1 on e_1 which is $\frac{1}{2}\epsilon$ distant away from the left end of e_1. If the vertical line, $v1$, which is 1 unit to the right from p_1, intersects with one horizontal edge at the bottom of $\tilde{P}$, then we choose this intersection as p_2. Otherwise we shift both p_1 and $v1$ to the right until $v1$ intersects with a straight edge at the bottom and then choose this new intersection as p_2. Since the radius of a round fixel is less than $\frac{1}{2}$, the horizontal projection of a round corner is less than $\frac{1}{2}$. Therefore the distance needed to shift to get p_2 is less than $\frac{1}{2}$. Therefore the initial choice of p_1 ensure that p_1 will be still on the straight edge e_1 after the shift. Now we pick up the upper most vertical edge, e_2, on the left side of $\tilde{P}$. Obviously e_2 is an edge belonging to either the case (1) or (2). Therefore e_2 also has a length $> \frac{1}{2} + \epsilon$ with $\epsilon > 0$. By the similar argument, we can find one point p_3 on e_2 and another point p_4 on a vertical straight edge on the right side of $\tilde{P}$, which is 1 unit below p_3. The four points p_1, p_2, p_3, p_4 define form closure. By zero rotation, we can always shift ∂P horizontally and vertically such that the four points defining the form closure are each vertically or horizontally close enough to a lattice site, at which we can anchor a fixel applying a force normal to ∂P at that point. □

6.3 Convex Fixturable Parts with Round Fixels

We can also generalize Theorem 3 to round fixels. In the expanded part, each edge segment of a convex polygonal part will be still a straight edge and keeps its original length. Furthermore the set of independent edge segments is still such a set in the expanded part because of the frictionless assumption. Therefore Theorem 3 still holds for round fixels.

7 Discussion

The notion of completeness is important in robot motion planning [Goldberg 1994]. We say that an algorithm is *complete* if it is guaranteed to find a solution when one exists, and to return failure otherwise. Completeness is desirable especially when algorithms are incorporated into industrial systems, where delays and failures can be extremely costly. The question we consider in this paper is stronger: under what conditions does a solution exist? We considered the existential question for 3L/1C and 0L/4C fixtures. We constructed an infinite set of convex polygonal parts that cannot be fixtured under the 3L/1C model. We also gave two subclasses of polygonal parts for which we can always guarantee the existence of a 0L/4C fixture.

The last result required that the critical length of a Nguyen set be greater than $\sqrt{2}$. We acknowledge that the Nguyen set for a part is not unique. Is there an algorithm that can efficiently verify this condition for a given polygonal part?

What about 3D fixtures? We note that the result for 3L/1C does not depend on the fact that the lattice is regular or rectangular. A similar argument would hold for any fixed arrangement of lattice sites. This approach can be extended to generate counterexamples for any modular fixturing system that requires 4 locators at 3D lattice sites.

In this paper we assumed zero friction. Friction can expand the range of parts that can be fixtured. For example we know that 3 contacts are often sufficient to provide frictional form closure. [Ponce 1993] showed how to find Nguyen sets of 3 segments for polygonal parts with frictional contacts. The existential question for modular fixtures with friction is currently open.

In summary, we have positive results for a general class of fixtures with a specific class of parts, and a negative result for a specific class of fixtures with a general class of parts. These results raise a number of questions about intermediate cases. For example, can we characterize a set of polygonal parts that are fixturable in the 3L/1C model? Similarly, can we characterize a set of polygonal parts that *are not* fixturable in the 0L/4C model? So far we have not been able to construct a single counterexample, although we are still unable to prove that 0L/4C is complete for the class of polygonal parts.

What about other fixture models? Recently, [Wallack and Canny 1994] studied a fixturing model using four locators on a split lattice that can be closed like a vice. Can we specify a class of parts for which such a fixture is guaranteed to exist? Similar questions remain for 2L/2C and 1L/3C fixture models.

These questions suggest a hierarchy of modular fixturing systems. We might say that modular fixturing system $\mathcal{A}$ *dominates* modular fixturing system $\mathcal{B}$ if the set of parts that can be fixtured in $\mathcal{A}$ is a strict superset of those that can be fixtured in $\mathcal{B}$. For example, we can show that 0L/4C dominates 3L/1C: For any part which has a 3L/1C fixture, we can always create a trivial 0L/4C fixture. Let P be the part constructed in Theorem 1 that is not fixturable in 3L/1C. We can easily insure that it has an odd number of edges and thus has no parallel sides. If we locate P on the lattice such that its center line is vertical, then we can place 4 clamps symmetrically about its center line. If the 4 normal vectors at contacts do not intersect at a common point, then this provides form closure. Otherwise, we can translate P vertically to break the common intersection to achieve form closure. Thus the class of 0L/4C fixtures dominates the 3L/1C class. Relations between other classes are currently open.

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References

[Asada and By 1985] H. Asada and Andre B. By. Kinematic analysis of workpart fixturing for flexible assembly with automatically reconfigurable fixtures. *IEEE Journal of Robotics and Automation*, RA-1(2), June 1985.

[Benson 1966] R. V. Benson. *Euclidian Geometry and Convexity*. McGraw-Hill Book Company, 1966.

[Brost and Goldberg 1994] Randy Brost and Ken Goldberg. A complete algorithm for synthesizing modular fixtures for polygonal parts. In *International Conference on Robotics and Automation*. IEEE, May 1994.

[Goldberg 1994] Ken Goldberg. Completeness in robot motion planning. In *Workshop on the Algorithmic Foundations of Robotics*, February 1994.

[Grupen, Henderson, and McCammon 1989] R. A. Grupen, T. C. Henderson, and I. D. McCammon. A survey of general purpose manipulation. *International Journal of Robotics Research*, 8(1), 1989.

[Hoffman 1987] E. G. Hoffman. *Modular Fixturing*. Manufacturing Technology Press, Lake Geneva, Wisconsin, 1987.

[Lyusternik 1966] L. A. Lyusternik. *Convex Figures and Polyhedra*, volume Survey of recent East European mathematical literature of *Topics in Mathematics*. D. C. Heath and Company, Boston, MA, June 1966.

[Markenscoff, Ni, and Papadimitriou 1990] Xanthippi Markenscoff, Luqun Ni, and Christos H. Papadimitriou. The geometry of grasping. *IJRR*, 9(1), February 1990.

[Mishra 1991] Bud Mishra. Workholding: Analysis and planning. In *International Conference on Intelligent Robots and Systems*. IEEE/RSJ, July 1991.

[Mishra, Schwartz, and Sharir 1987] Bud Mishra, J. T. Schwartz, and M. Sharir. On the existence and synthesis of multifinger positive grips. *Algorithmica*, 2(4):641–558, 1987.

[Nguyen 1988] Van-Duc Nguyen. Constructing force-closure grasps. *International Journal of Robotics Research*, 7(3), 1988.

[Ohwovoriole 1987] E. N. Ohwovoriole. Kinematics and friction in grasping by robotic hands. *Transactions of the ASME*, 109, 1987.

[Overmars *et al.*] Mark Overmars, Anil Rao, Otfried Schwarzkopf, and Chantal Wentink. Immobolizing polygons against a wall. In *ACM Symposium on Computational Geometry*, 1995. Vancouver, BC.

[Penev and Aristides 1995] Kamen Penev and Aristides A. G. Requicha. Fixture foolproofing for polygonal parts. In *International Symposium on Assembly and Task Planning*. IEEE, August 1995. Pittsburgh, PA.

[Pertin-Troccaz 1989] Jocelyn Pertin-Troccaz. Grasping: A state of the art. In *The Robotics Review I*, pages 71–98. MIT Press, 1989. edited by O. Khatib, J. J. Craig, and T. Lozano-Perez.

[Ponce 1993] Jean Ponce and Bernard Faverjon. On computing three-finger force-closure grasps of polygonal objects. Technical Report UIUC-BI-AI-RCV-93-05, Beckman Institute, University of Illinois, 1993.

[Requicha 1980] Ari A. G. Requicha. Representations for rigid solids: Theory, models, and systems. *ACM Computing Surveys*, 12:437–464, 1980.

[Reuleaux 1876] F. Reuleaux. *The Kinematics of Machinery*. Macmilly and Company, 1876. Republished by Dover in 1963.

[Trinkle 1992] Jeff C. Trinkle. A quantitative test for form closure grasps. In *International Conference on Intelligent Robots and Systems*. IEEE/RSJ, July 1992.

[Wallack and Canny 1994] Aaron Wallack and John Canny. Planning for modular and hybrid fixtures. In *International Conference on Robotics and Automation*. IEEE, May 1994.

[Yaglom and Boltyanskii 1961] I. M. Yaglom and V.G. Boltyanskii. *Convex Figures*. Holt, Rinehart and Winston, New York, 1961.

[Zhuang, Goldberg, and Wong 1994] Yan Zhuang, Ken Goldberg, and Yin-Chung Wong. On the existence of modular fixtures. In *International Conference on Robotics and Automation*. IEEE, May 1994.