IEOR 265 – Lecture 12
Reachability

1 Lyapunov Stability

Consider an autonomous nonlinear dynamical system in discrete time

$$x_{n+1} = f(x_n), \quad x_0 = \xi$$

with an equilibrium point at $x^* = 0$. Some important types of stability that we have covered are:

- A system is *Lyapunov stable* if given $M_2 > 0$ there exists $M_1 > 0$ such that $\|x_0\| \leq M_1$ implies that $\|x_n\| \leq M_2$ for all $n \geq 0$.
- A system is *locally asymptotically stable* (LAS) if (a) it is Lyapunov stable, and (b) there exists $M_3 > 0$ such that $\|x_n\| \to 0$ whenever $\|x_0\| \leq M_3$. A system is *globally asymptotically stable* (GAS) if $M_3 = \infty$.
- A system is *exponentially stable* if (a) it is asymptotically stable, and (b) there exists $M_3 > 0$ and $\alpha, \beta > 0$ such that $\|x_n\| \leq \alpha \|x_0\| \exp(-\beta n)$ whenever $\|x_0\| \leq M_3$. A system is *globally exponentially stable* if $M_3 = \infty$.

2 Lyapunov Function

Our tests for stability have been focused on discrete time LTI systems, and so it is natural to ask how to show stability for a nonlinear system in discrete time. To do so, we must first give some abstract definitions

- A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is type-$\mathcal{K}$ if it is continuous, strictly increasing, and $\gamma(0) = 0$.
- A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{R}_+$ is type-$\mathcal{KL}$ if for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is type-$\mathcal{K}$, and for each fixed $s \geq 0$ the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

With these definitions, we can now define a time-varying function that will indirectly allow us show that a system is stable. The function $V_n : \mathcal{X} \to \mathbb{R}$ is a Lyapunov function for a discrete time system if the following conditions hold:

1. $V_n(0) = 0$ and $V_n(x) > 0$ for all $x \neq 0$;
2. $\alpha_1(\|x\|) \leq V_n(x) \leq \alpha_2(\|x\|)$, where α_1, α_2 are type-$\mathcal{K}$ functions;
3. $x \in \text{int}(\mathcal{X})$, that is x is in the interior of $\mathcal{X}$;

4. $V_n(x_{n+1}) - V_n(x_n) \leq 0$;

The intuition of the first three conditions is that the Lyapunov function V_n is like an energy function that is zero at the equilibrium $x^* = 0$ and increases in value as it gets farther from the equilibrium. The last condition says that the value of the Lyapunov function evaluated at the current state of the system is non-increasing, and the intuition is that the energy of the system does not increase.

3 Lyapunov Theorems

These definitions lead to several theorems. In particular, we have that if an autonomous nonlinear discrete time system has:

- a Lyapunov function (as defined above with $V_n(x_n + 1) - V_n(x_n) \leq 0$), then then system is Lyapunov stable;
- a Lyapunov function such that $V_n(x_{n+1}) - V_n(x_n) < 0$ (that is strictly decreasing) for $x_n \neq 0$, then the system is LAS;
- a Lyapunov function such that $V_n(x_{n+1}) - V_n(x_n) < 0$ (that is strictly decreasing) for $x_n \neq 0$ and the domain of the Lyapunov function is $\mathcal{X} = \mathbb{R}^p$, then the system is GAS.
- a Lyapunov function such that $V_n(x_{n+1}) - V_n(x_n) < -\alpha V_n(x_n)$ for $x_n \neq 0$, V_n is such that $\alpha_1(\|x\|) = \kappa_1 \cdot \|x\|$ and $\alpha_2(\|x\|) = \kappa_2 \cdot \|x\|$, for some fixed $\alpha, \kappa_1, \kappa_1 > 0$, then the system is exponentially stable.
- a Lyapunov function such that $V_n(x_{n+1}) - V_n(x_n) < -\alpha V_n(x_n)$ for $x_n \neq 0$, V_n is such that $\alpha_1(\|x\|) = \kappa_1 \cdot \|x\|$ and $\alpha_2(\|x\|) = \kappa_2 \cdot \|x\|$, for some fixed $\alpha, \kappa_1, \kappa_1 > 0$, and the domain of the Lyapunov function is $\mathcal{X} = \mathbb{R}^p$, then the system is globally exponentially stable.

Note that we cannot use the above theorems to show that a system is unstable. This is in fact the biggest weakness of Lyapunov theory: There is no systematic way to compute a Lyapunov function for a system, unless the system is linear (or has polynomial dynamics).

4 Examples

We have already seen some examples of Lyapunov functions, specifically for LTI systems. Here, we recall the past examples and give a new example:

- Note that a discrete time system is (exponentially and asymptotically) stable if and only there exists $P > 0$ such that $A'PA - P < 0$, or equivalently given any $Q > 0$ there exists $P > 0$ such that $A'PA - P = -Q$. In this case, a Lyapunov function is $V(x) = x'Px$.

- Another example occurs in the infinite horizon LQR case. Consider the value function of the optimization

$$V(x_0) = \min \left\{ \sum_{n=0}^{\infty} x_n' Q x_n + u_n' R u_n : x_{n+1} = A x_n + B u_n; x_0 = \xi \right\},$$

where $Q > 0$ and $R > 0$ are positive definite matrices and (A, B) is stabilizable. The value function is equal to $V(x_0) = x_0' P x_0$ where $P > 0$ is the unique solution to the discrete time algebraic Riccati equation (DARE)

$$P = Q + A'(P - PB(R + B'PB)^{-1} B'P)A.$$

For the state-feedback of $u_n = K x_n$ where $K = -(R + B'PB)^{-1} B'PA$, the value function of this optimization problem is a Lyapunov function for the closed-loop system $x_{n+1} = (A + BK)x_n$. In fact, a straightforward calculation gives

$$\begin{aligned}
&(A + BK)'P(A + BK) - P \\
&\qquad = (A + BK)'P(A + BK) - P \\
&\qquad = A'PA + K'B'PA + A'PBK + K'B'PBK - P \\
&\qquad = A'PA + A'PBK + K'B'PA + K'(R + B'PB)K - K'RK - P \\
&\qquad = A'PA + A'PBK + K'B'PA - K'B'PA - K'RK \\
&\qquad = A'PA + A'PBK - P - K'RK
\end{aligned}$$

but the DARE can be rewritten as $P = Q + A'PA + A'PBK$, and so we have that

$$(A + BK)'P(A + BK) - P = A'PA + A'PBK - P - K'RK = -Q - K'RK < 0.$$

- As a final example, consider the following autonomous nonlinear system

$$x_{n+1} = x_n^2,$$

where $x_n, x_{n+1} \in \mathbb{R}$. If we choose $V(x) = x^2$, then we have that

$$V(x_{n+1}) - V(x_n) = x_n^4 - x_n^2 = x_n^2(x_n^2 - 1).$$

If $x_n^2 - 1 < 0$ (or equivalently $-1 < x_n < 1$), then $V(x_{n+1}) - V(x_n) < 0$. This choice of V satisfies the criterion for being a Lyapunov function for this particular system. Note that if $|x_n| > 1$ then the system is not stable in any sense.

5 Polytope Constraints

So far, we have not considered constraints on our input u_n or states x_n, when designing feedback controllers for LTI systems in discrete-time; however, there are many applications in which we would like to impose constraints on both the inputs and states. In some sense, though the constraints often arise from physical or economic limitations on the system being modeled, we get to design the constraints that we use for engineering. Specifically, we can choose any representation of constraints that respects the physical or economic limitations of the system, even if our constraints are more conservative than needed. Such choices can be necessary for the purpose of efficient computation.

In fact, a fairly broad class of constraints with useful computational and mathematical properties can be defined by *bounded convex polytopes*, which can be defined as the convex hull of the set of points. Note that when we refer to polytopes in the future, we will specifically mean bounded convex polytopes. The reason that polytopes are an attractive approach to defining constraints is that they can be represented as the intersection of half-spaces. Recall that a half-space can be represented by $f_i'x \leq h_i$, and so the intersection of half-spaces can be represented by multiple linear inequalities: $Fx \leq h$.

So if we have a polytope $\mathcal{X} = \{x : Fx \leq h\}$, then the constraint that $x_n \in \mathcal{X}$ means that we would like x_n such that $Fx_n \leq h$. We will often refer to constraints on the states and inputs by referring to the polytopes in which they lie; that is, we will ask that $x_n \in \mathcal{X}$ and $u_n \in \mathcal{U}$, where $\mathcal{X}, \mathcal{U}$ are polytopes.

5.1 Box Constraints

A common type of constraint are *box constraints*. For a vector $x_n \in \mathbb{R}^p$, a box constraint is that there exists a_i, b_i for $i = 1, \ldots, p$ such that $a_i \leq x_n^i \leq b_i$ for all i. It turns out that we can express these constraints as a polytope:

$$x_n \in \mathcal{X} = \{x : x_i \leq b_i, -x_i \leq -a_i, \forall i\}.$$

5.2 Linear Transform of Polytope

We define the linear transform T of a polytope $\mathcal{P} = \{u : Fu \leq h\}$ as the polytope

$$T\mathcal{P} = \{Tu : u \in \mathcal{P}\}.$$

6 Maximal Output Invariant Sets

Consider an LTI system in discrete time:

$$x_{n+1} = Ax_n + Bu_n,$$

where (A, B) is stabilizable. And assume that we have chosen a K such that using the state-feedback controller $u_n = Kx_n$ leads to a stable system $x_{n+1} = (A + BK)x_n$. Now consider this same system, and suppose that we have polytopic constraints: In particular, we require that

$x_n \in \mathcal{X}$ and $u_n \in \mathcal{U}$ for all $n \geq 0$. A natural question to ask is: Does there exist a set Ω such that if $x_0 \in \Omega$, then the controller $u_n = Kx_n$ ensures that both constraints are satisfied. In mathematical terms, we would like this set Ω to achieve (a) constraint satisfaction

$$\Omega \subseteq \{x : x \in \mathcal{X}; Kx \in \mathcal{U}\},$$

and (b) control invariance

$$(A + BK)\Omega \subseteq \Omega.$$

It can be shown that if $0 \in \mathcal{X}$ and $0 \in \mathcal{U}$, then the set Ω can be represented by a polytope with a finite number of constraints. There is also an algorithm to compute this set:

input : $X = \{x : F_x x \leq h_x\}$ and $U = \{u : F_u u \leq h_u\}$
input : A, B, K
output: Ω

set $t \leftarrow 0$;
set $k_1 \leftarrow \text{rows}(h_x)$;
set $k_u \leftarrow \text{rows}(h_u)$;
repeat
  for $j \leftarrow 1$ **to** k_1 **do**
    set $L_j^* \leftarrow \max\{(F_x)_j(A + BK)^{t+1}x - (h_x)_j : (A + BK)^k x \in \mathcal{X}, K(A + BK)^k x \in \mathcal{U}, \forall k = 0, \dots, t\}$;
  end
  for $j \leftarrow 1$ **to** k_2 **do**
    set $M_j^* \leftarrow \max\{(F_u)_j K(A + BK)^{t+1}x - (h_u)_j : (A + BK)^k x \in \mathcal{X}, K(A + BK)^k x \in \mathcal{U}, \forall k = 0, \dots, t\}$;
  end
  set $t \leftarrow t + 1$;
until $L_i^* \leq 0, \forall i = 1, \dots, k_1$ *and* $M_i^* \leq 0, \forall i = 1, \dots, k_2$;
set $t^* \leftarrow t - 1$;
set $\Omega = \{x : (A + BK)^k x \in \mathcal{X}, K(A + BK)^k x \in \mathcal{U}, \forall k = 0, \dots, t^*\}$;

Note that the Ω returned by this algorithm is a polytope, and so we can rearrange terms to express this set as $\Omega = \{x : F_\omega x \leq h_\omega\}$.