IEOR 265 – Lecture 11
Parametric Optimization

1 Extended-Real-Valued Functions

A common formulation of a finite-dimensional optimization problem is

$$\begin{aligned} \min\ & f(x) \\ \text{s.t. } & g_i(x) \leq 0, \forall i = 1, \ldots, I_1 \\ & h_i(x) = 0, \forall i = 1, \ldots, I_2 \\ & x \in \mathcal{X} \subseteq \mathbb{R}^p \end{aligned}$$

where $f(x), g_i(x), h_i(x)$ are functions that have a domain that is a subset of $\mathbb{R}^p$, and $f(x)$ is a function with domain in $\mathbb{R}$. It turns out that for certain applications, it can be useful to redefine this optimization using extended-real-valued functions.

The extended-real-valued line is defined as $\overline{\mathbb{R}} = [-\infty, \infty]$ (compare this to the real-valued line $\mathbb{R} = (-\infty, \infty)$). The difference between these two lines is that extended-real-valued line specifically includes the values $-\infty$ and ∞, whereas these are not numbers in the real-valued line.

The reason that this concept is useful is that it can be used to reformulate the above optimization problem. In particular, suppose that we define an extended-real-valued function $\tilde{f}$ as follows

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } g_i(x) \leq 0, \forall i = 1, \ldots, I_1; h_i(x) = 0, \forall i = 1, \ldots, I_2; x \in \mathcal{X} \subseteq \mathbb{R}^p \\ \infty, & \text{otherwise} \end{cases}$$

We can hence formulate the above optimization problem as the following unconstrained optimization

$$\min \tilde{f}(x).$$

It is worth emphasizing this point: One benefit of formulating optimization problems using extended-real-valued functions is that it allows us to place the constraints and objective on equal footing.

2 Epigraph

An important concept in variational analysis is that of the epigraph. In particular, suppose we have an optimization problem

$$\min f(x),$$

where $f : \mathbb{R}^p \to \overline{\mathbb{R}}$ is an extended-real-valued function. We define the *epigraph* of f to be the set

$$\text{epi } f = \{(x, \alpha) \in \mathbb{R}^p \times \mathbb{R} \mid \alpha \geq f(x)\}.$$

Note that the epigraph of f is a subset of $\mathbb{R}^p \times \mathbb{R}$ (which does not include the extended-real-valued line).

3 Lower Semicontinuity

We define the *lower limit* of a function $f : \mathbb{R}^p \to \overline{\mathbb{R}}$ at $\overline{x}$ to be the value in $\overline{\mathbb{R}}$ defined by

$$\liminf_{x \to \overline{x}} f(x) = \lim_{\delta \searrow 0} \left[\inf_{x \in \mathcal{B}(\overline{x}, \delta)} f(x) \right] = \sup_{\delta > 0} \left[\inf_{x \in \mathcal{B}(\overline{x}, \delta)} f(x) \right],$$

where $\mathcal{B}(\overline{x}, \delta)$ is a ball centered at $\overline{x}$ with radius δ. Similarly, we define the *upper limit* of f at $\overline{x}$ as

$$\limsup_{x \to \overline{x}} f(x) = \lim_{\delta \searrow 0} \left[\sup_{x \in \mathcal{B}(\overline{x}, \delta)} f(x) \right] = \inf_{\delta > 0} \left[\sup_{x \in \mathcal{B}(\overline{x}, \delta)} f(x) \right].$$

We say that the function $f : \mathbb{R}^p \to \overline{\mathbb{R}}$ is lower semicontinuous (lsc) at $\overline{x}$ if

$$\liminf_{x \to \overline{x}} f(x) \geq f(\overline{x}), \text{or equivalently } \liminf_{x \to \overline{x}} f(x) = f(\overline{x}).$$

Furthermore, this function is lower semicontinuous on $\mathbb{R}^p$ if the above condition holds for every $\overline{x} \in \mathbb{R}^p$. There are some useful characterizations of lower semicontinuity:

- the epigraph set epi f is closed in $\mathbb{R}^p \times \mathbb{R}$;
- the level sets of type $\text{lev}_{\leq a} f$ are all closed in $\mathbb{R}^p$.

One reason that lower semicontinuity is important is that if f is lsc, level-bounded (meaning the level sets $\text{lev}_{\leq a} f$ are bounded), and proper (meaning that the preimage of every compact set is compact), then the value $\inf f$ is finite and the set $\arg\min f$ is nonempty and compact. This means that we can replace $\inf f$ by $\min f$ in this case.

4 Definitions

Many problems can be cast as a parametric optimization problem. These are optimization problems in which there is a free parameter $\theta \in \mathbb{R}^p$, and the optimization to be solved is different depending on the value of θ. Notationally, we write

$$\begin{aligned} V(\theta) = \min \ & f(x; \theta) \\ \text{s.t. } & x \in C(\theta). \end{aligned}$$

We will refer to $V(\theta)$ as the value function. Similarly, we will write

$$x^*(\theta) = \arg\min\ f(x;\theta)$$
$$\text{s.t. } x \in C(\theta).$$

The $x^*(\theta)$ are the minimizers. These types of problems are common in game theory, control engineering, and mathematical economics. One important question is whether the minimizers $x^*(\theta)$ and value function $V(\theta)$ are continuous with respect to the parameter θ.

5 Hemicontinuity

We cannot use the normal definitions of continuity because the minimizers $x^*(\theta)$ may not be single-valued, meaning that there may be zero, one, multiple, or infinitely many minimizers. The normal definitions of continuity do not apply in this case. This is also the situation for the constraints $C(\theta)$ which will in general be a set that depends on θ. Because we have multivalued functions, we must define a new type of continuity. In fact, the concept of hemicontinuity is the correct generalization for our purposes.

5.1 Upper Hemicontinuity

We say that $C(\cdot) : \mathbb{R}^p \to \mathbb{R}^n$ is upper hemicontinuous at $\theta \in \mathbb{R}^p$ if $\forall \theta_k \in \mathbb{R}^p$, $\forall x_k \in C(\theta_k)$, and $\forall x \in \mathbb{R}^n$; we have

$$\lim_{k\to\infty} \theta_k = \theta, \lim_{k\to\infty} x_k \to x \Rightarrow x \in C(\theta).$$

The intuition is that every sequence $x_k \in C(\theta_k)$ converges to a point $x \in C(\theta)$.

5.2 Lower Hemicontinuity

We say that $C(\cdot) : \mathbb{R}^p \to \mathbb{R}^n$ is lower hemicontinuous at $\theta \in \mathbb{R}^p$ if $\forall \theta_k \in \mathbb{R}^p$ such that $\theta_k \to \theta$, $\forall x \in C(\theta)$, there exists a subsequence θ_{k_j} and $x_k \in C(\theta_{k_j})$ such that $x_k \to x$. The intuition is that for every point $x \in C(\theta)$ there is a sequence $x_k \in C(\theta_k)$ that converges to the point.

5.3 Examples

There are two important examples of constraint sets for which $C(\theta)$ is both upper and lower hemicontinuous. The first example is when the constraint set is independent of θ. The second example is when the constraint set is linear in both x and θ.

6 Berge Maximum Theorem

The Berge Maximum Theorem provides an interesting set of conditions for when the minimizers and value function are appropriately continuous. The reason for stating appropriately continuous is that the function $C(\theta)$ is not a function in the normal sense; it is a multivalued function, and so continuity has to be redefined as above. The intuition of the theorem is that if the objective is continuous in the usual way, and the constraint set is continuous in an appropriate way, then the value function is continuous and the minimizer is continuous in an appropriate way. More formally:

If $f(x;\theta)$ is jointly continuous in x, θ and $C(\theta)$ is compact-valued, lower hemicontinuous, and upper hemicontinuous, then the value function $V(\theta)$ is continuous and the minimizer $x^*(\theta)$ is nonempty, compact-valued, and upper hemicontinuous.

There is an important note: Upper hemicontinuity at a point that is single-valued means that the function is continuous (in the traditional sense) at that point. As a result, if the minimizer $x^*(\theta)$ is single-valued, then the function is continuous at $x^*(\theta)$.

6.1 Variants

If the objective $f(x,\theta)$ is convex in (x,θ) and the graph of $C(\theta)$ (meaning the set $\{(x,\theta) : x \in C(\theta)\}$) is convex, then the value function $V(\theta)$ is convex. Furthermore, if $f(x,\theta)$ is strictly convex in (x,θ) and the graph of $C(\theta)$ is convex, then the value function $V(\theta)$ is convex and the minimizer $x^*(\theta)$ is single-valued and continuous in θ. Note that in general for both situations, $x^*(\theta)$ will not be convex in θ.

7 Further Details

More details about these variational analysis concepts can be found in the book *Variational Analysis* by Rockafellar and Wets, from which the above material is found.