IEOR 265 – Lecture 16
Learning-Based MPC

1 Robustness in Tube Linear MPC

1.1 Continuity of Value Function

In addition to the recursive feasibility and constraint satisfaction properties of tube MPC, in the presence of disturbance, there are other types of robustness that this method has. These are in fact related to continuity of the minimizers and value function. In fact, if ψ_n is continuous and strictly convex, then the strictly convex variant of the Berge maximum theorem tells us that the minimizer is continuous and the value function is convex and continuous.

1.2 Robust Asymptotic Stability

Recall the following two definitions:

- A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is type-$\mathcal{K}$ if it is continuous, strictly increasing, and $\gamma(0) = 0$.
- A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{R}_+$ is type-$\mathcal{KL}$ if for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is type-$\mathcal{K}$, and for each fixed $s \geq 0$ the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

We can now give our main definition for this section: A system is *robustly asymptotically stable (RAS)* if there exists a type-$\mathcal{KL}$ function β and for each $\epsilon > 0$ there exists $\delta > 0$, such that for all d_n satisfying $\max_n \|d_n\| < \delta$ it holds that $x_n \in \mathcal{X}$ and $\|x_n\| \leq \beta(\|x_0\|, n) + \epsilon$ for all $n \geq 0$.

It turns out that the tube MPC formulation has RAS when the objective is given by

$$\psi_n = \overline{x}'_{n+N} P \overline{x}_{n+N} + \sum_{k=0}^{N-1} \overline{x}'_{n+k} Q \overline{x}_{n+k} + \check{u}_{n+k} R \check{u}_{n+k},$$

where $(A + BK)'P(A + BK) - P = -(Q + K'RK)$. The proof is somewhat technical and will not be covered here. The key feature of the proof is that there is a continuous Lyapunov function for the nominal system, because the value function is continuous. And because the Lyapunov function shows sufficient descent, small perturbations will not remove descent except for some region about the origin.

2 Optimization Formulation of Learning-Based MPC

Recall the LTI system with modeling error

$$x_{n+1} = Ax_n + Bu_n + g(x_n, u_n),$$

where $g(x_n, u_n) \in \mathcal{W}$ for all $x_n \in \mathcal{X}$,$u_n \in \mathcal{U}$. Previously, we have treated this modeling error as disturbance; however, this can be overly conservative because as we control the system, we get more information about the system that we can use to update our model of the system. This is the purpose of learning-based MPC (LBMPC).

The main technical challenge of incorporating adaptation or machine learning into control and optimization frameworks is that of ensuring robustness, and the key insight of LBMPC is that this can be achieved by maintaining two distinct models of the system; this leads to a novel formulation as well as a novel approach for designing general robust adaptive optimization methods. We define the LBMPC optimization formulation as

$$V_n(x_n) = \min\nolimits_c \psi_n(\tilde{x}_n, \ldots, \tilde{x}_{n+N}, \check{u}_n, \ldots, \check{u}_{n+N-1}) \tag{1}$$

subject to:

$$\tilde{x}_n = x_n, \quad \overline{x}_n = x_n \tag{2}$$

$$\tilde{x}_{n+i+1} = A\tilde{x}_{n+i} + B\check{u}_{n+i} + \mathcal{O}_n(\tilde{x}_{n+i}, \check{u}_{n+i}) \tag{3}$$

$$\left.\begin{aligned} &\overline{x}_{n+i+1} = A\overline{x}_{n+i} + B\check{u}_{n+i} \\ &\check{u}_{n+i} = K\overline{x}_{n+i} + c_{n+i} \\ &\overline{x}_{n+i+1} \in \mathcal{X} \ominus \mathcal{R}_{i+1}, \quad \check{u}_{n+i} \in \mathcal{U} \ominus K\mathcal{R}_i \\ &\overline{x}_{n+N} \in \Omega \ominus \mathcal{R}_N \end{aligned}\right\} \tag{4}$$

for all $i = 0, \ldots, N-1$ in the constraints; K is the feedback gain used to compute Ω; $\mathcal{R}_0 = \{0\}$ and $\mathcal{R}_i = \bigoplus_{j=0}^{i-1}(A+BK)^j\mathcal{W}$; $\mathcal{O}_n$ is the oracle; and ψ_n are non-negative functions that are Lipschitz continuous in their arguments. The idea of the oracle $\mathcal{O}_n$ is that it is a function that given a new value of x_n and u_n it returns an estimate of the modeling error (and potentially a gradient of this estimate).

3 Recursive Properties

Suppose that (A, B) is stabilizable and K is a matrix such that $(A + BK)$ is stable. For this given system and feedback controller, suppose we have a maximal output admissible disturbance invariant set Ω (meaning that this set has constraint satisfaction $\Omega \subseteq \{x : x \in \mathcal{X}, Kx \in \mathcal{U}\}$ and disturbance invariance $(A + BK)\Omega \oplus \mathcal{W} \subseteq \Omega$.

Next, note that conceptually our decision variables are c_{n+i} since the $\overline{x}_k$,$\tilde{x}_k$,$\check{u}_k$ are then uniquely determined by the initial condition and equality constraints. As a result, we will talk about solutions only in terms of the variables c_k. In particular, if $\mathcal{M}_n = \{c_n, ..., c_{n+N1}\}$ is feasible for the optimization defining LBMPC with an initial condition x_n, then the system that applies the control value $u_n = Kx_n + c_n[M_n]$ results in:

- Recursive Feasibility: there exists feasible $\mathcal{M}_{n+1}$ for x_{n+1};
- Recursive Constraint Satisfaction: $x_{n+1} \in \mathcal{X}$.

The proof is actually identical to that for linear tube MPC, because the learned model with oracle are not constrained by $\mathcal{X}$ or $\mathcal{U}$. There is the same collorary of this theorem: If there exists feasible $\mathcal{M}_0$ for x_0, then the system is (a) Lyapunov stable, (b) satisfies all state and input constraints for all time, (c) feasible for all time.

4 Continuity of Value Function

In addition to the recursive feasibility and constraint satisfaction properties of LBMPC, in the presence of disturbance, there are other types of robustness that this method has. These are in fact related to continuity of the minimizers and value function. In fact, if ψ_n and $\mathcal{O}_n$ continuous, then the Berge maximum theorem tells us that the value function is continuous.

5 Robust Asymptotic Stability

Recall the following definition: A system is robustly asymptotically stable (RAS) if there exists a type-$\mathcal{KL}$ function β and for each $\epsilon > 0$ there exists $\delta > 0$, such that for all d_n satisfying $\max_n \|d_n\| < \delta$ it holds that $x_n \in \mathcal{X}$ and $\|x_n\| \leq \beta(\|x_0\|, n) + \epsilon$ for all $n \geq 0$.

Suppose Ω is a maximal output admissible disturbance invariant set, $\mathcal{M}_0$ is feasible for x_0, and the cost function is

$$\psi_n = \tilde{x}'_{n+N} P \tilde{x}_{n+N} + \sum_{k=0}^{N-1} (\tilde{x}'_{n+k} Q \tilde{x}_{n+k} + \check{u}'_{n+k} R \check{u}_{n+k}),$$

where $(A + BK)' P (A + BK) - P = -(Q + K' R K)$, and lastly that $\mathcal{O}_n$ is a continuous function satisfying $\max_{n, \mathcal{X} \times \mathcal{U}} \|\mathcal{O}_n\| \leq \delta$. Under these conditions, LBMPC is also RAS. The proof is technical, and the basic idea is that we compare the solution of LBMPC to the solution of linear MPC (i.e., $\mathcal{O}_n \equiv 0$). Interestingly, the proof makes use of the continuity of the minimizer for linear MPC.