IEOR 265 – Lecture 10
Stability

1 Nonlinear Dynamical Systems

Consider the following nonlinear dynamical system in discrete time:

$$x_{n+1} = f(x_n, u_n), \quad x_0 = \xi$$

where $x_n, x_{n+1} \in \mathbb{R}^p$, $u_n \in \mathbb{R}^q$, $f : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}$ is a nonlinear function, the subscripts denote discrete time indices, and $\xi \in \mathbb{R}^p$ is an initial condition. In this model, the x_n are called *states*, and the u_n are *inputs*. The states represent the internal memory of the system, and they also summarize the history (past states and inputs) of the system. The inputs can be used to steer the system to satisfy engineering objectives.

A point x^* is an *equilibrium point* of the nonlinear system if there exists u^* such that $x^* = f(x^*, u^*)$. The intuition is that this is a point such that a constant input of u^* keeps the system at x^* whenever the system is already at x^*. Without loss of generality, we will assume that $x^* = 0$ and $u^* = 0$; note that this also implies that $f(0, 0) = 0$. (We can always define a translation of our coordinate system in which this assumption is true.)

2 Definitions of Stability

Such nonlinear systems have many interesting properties, and we will begin by discussing one type of property known as stability. In fact, there are several notions of stability in nonlinear systems, and the intuition is that a stable system has states that remain bounded.

2.1 Lyapunov Stability

A system is Lyapunov stable if given $M_2 > 0$ there exists $M_1 > 0$ such that $\|x_0\| \leq M_1$ implies that $\|x_n\| \leq M_2$ for all $n \geq 0$.

2.2 Asymptotic Stability

We say that a system is locally asymptotically stable (LAS) if (a) it is Lyapunov stable, and (b) there exists $M_3 > 0$ such that $\|x_n\| \to 0$ whenever $\|x_0\| \leq M_3$. A system is globally asymptotically stable (GAS) if $M_3 = \infty$.

2.3 Exponential Stability

A system is exponentially stable if (a) it is asymptotically stable, and (b) there exists $M_3 > 0$ and $\alpha, \beta > 0$ such that $\|x_n\| \leq \alpha \|x_0\| \exp(-\beta n)$ whenever $\|x_0\| \leq M_3$. A system is globally exponentially stable if $M_3 = \infty$.

3 Linear Dynamical Systems

An important special case of a nonlinear dynamical system is the situation in which the dynamics are linear and time invariant. In this case, the linear time invariance (LTI) dynamical system in discrete time can be described by

$$x_{n+1} = Ax_n + Bu_n, \quad x_0 = \xi$$

where $A \in \mathbb{R}^{p\times p}$ and $B \in \mathbb{R}^{p\times q}$. This special case is useful because it can be analyzed using powerful machinery from linear algebra.

The first thing to observe is that we can identify every equilibrium point of this linear system. Note that equilibrium points satisfy

$$x^* = Ax^* + Bu^* \Rightarrow \begin{bmatrix} \mathbb{I} - A & B \end{bmatrix} \begin{bmatrix} x^* \\ -u^* \end{bmatrix} = 0.$$

Thus, the set of all equilibrium points and the corresponding control inputs is characterized by the null space of the matrix $\begin{bmatrix} \mathbb{I} - A & B \end{bmatrix}$.

Stability is relatively straightforward for LTI systems. It turns out that LAS, GAS, and exponential stability are equivalent for LTI systems. Furthermore, LAS/GAS/exponential stability imply Lyapunov stability. Given a matrix A, there are multiple ways to check for exponential stability. An LTI system is exponentially stable if and only if

1. $|\sigma(A)| < 1$ (all eigenvalues have magnitude strictly less than one);
2. $\mathrm{rank}(\begin{bmatrix} s\mathbb{I} - A \end{bmatrix}) = p$ for all $s \in \mathbb{C} : |s| \geq 1$;
3. given any $Q > 0$, there exists unique $P > 0$ such that $A'PA - P < -Q$; this equation is known as a Lyapunov equation, and it is a Linear Matrix Inequality (LMI) which means that it can be represented as a convex feasibility problem; the interpretation is that $x'Px$ is an energy function that decreases at rate

$$x'_{n+1}Px_{n+1} - x'_nPx_n = x'_nA'PAx_n - x'_nPx_n = x'_n(A'PA - P)x_n < -x'_nQx_n;$$

3.1 Bounded-Input Bounded-Output Stability

An LTI system is bounded-input bounded-output (BIBO) stable if there exists $0 \leq k < \infty$ such that for all bounded input sequences $\{u_0, u_1, \ldots\}$ the following holds:

$$\max_{n\geq 0} \|x_n\| \leq k \cdot \max_{n\geq 0} \|u_n\|.$$

One can think of the k as a gain. It turns out that BIBO stability is equivalent to LAS/GAS/exponential stability for LTI systems.

4 Controllability

Consider a discrete time LTI system:

$$x_{n+1} = Ax_n + Bu_n, \quad x_0 = \xi.$$

We have two related definitions. The LTI system defined by the pair (A, B) is:

1. *controllable* if and only if given any time $m \geq p + 1$ and any coordinate ϕ there exists a sequence of inputs $u_0, u_1, \ldots, u_{m-1}$ such that $x_m = \phi$;

2. *stabilizable* if and only if there exists a sequence of inputs $u_0, u_1, \ldots$ such that $\|x_n\| \to 0$.

These definitions are related because if an LTI system is controllable, then it is also stabilizable. The converse is not true: There are stabilizable LTI systems that are not controllable.

5 Conditions

We will describe several conceptual approaches to checking for controllability or stabilizability for (A, B).

1. Define the *controllability matrix*

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \ldots & A^{p-1}B \end{bmatrix}$$

   The pair (A, B) is controllable if and only if $\text{rank}(\mathcal{C}) = p$.

2. The *PopovBelevitchHautus* (PBH) test is that (A, B) is controllable if and only if

$$\text{rank}\left(\begin{bmatrix} s\mathbb{I} - A & B \end{bmatrix}\right) = p, \forall s \in \mathbb{C}.$$

   Furthermore, (A, B) is stabilizable if and only if

$$\text{rank}\left(\begin{bmatrix} s\mathbb{I} - A & B \end{bmatrix}\right) = p, \forall s \in \mathbb{C} : |s| \geq 1.$$

3. Consider matrices A such that that $|(\sigma(A))_i| < 1$. The pair (A, B) is controllable if and only if the unique solution W to

$$AWA' - W = -BB'$$

   is positive definite (i.e., $W > 0$). Observe that this is an LMI and can be solved using convex optimization approaches. Note that this W (if it exists) is equal to

$$W = \sum_{k=0}^{\infty} A^k BB'(A')^k,$$

   which is known as the reachability Gramian.

4. The pair (A, B) is stabilizable if and only if there is a positive definite $P > 0$ solution to

$$APA' - P < BB'.$$

   Note that this is an LMI and can be solved using convex optimization approaches.

6 Linear Feedback

The concepts of controllability and stabilizability are important because of the following result: An LTI system (A, B) is stabilizable if and only if there exists a constant matrix $K \in \mathbb{R}^{p \times q}$ such that choosing state-feedback input $u = Kx$ leads to a stable system

$$x_{n+1} = Ax_n + Bu_n = Ax_n + BKx_n = (A + BK)x_n,$$

meaning that the eigenvalues of $A + BK$ lie within the complex unit disc.

The condition of controllability is even more powerful. Let $\lambda_1, \lambda_2, \ldots, \lambda_p \in \mathbb{C}$ be fixed complex numbers. If (A, B) is controllable, then there exists a K such that the eigenvalues of $A + BK$ are precisely the $\lambda_1, \lambda_2, \ldots, \lambda_p$ that were chosen.

7 Finite Horizon Linear Quadratic Regulator (LQR)

Consider the following optimization problem

$$\min \left\{ \sum_{n=0}^{N} x_n' Q x_n + u_n' R u_n : x_{n+1} = Ax_n + Bu_n \right\},$$

where $Q > 0$ and $R > 0$ are positive definite matrices. The minimizer is given by $u_n = K_n x_n$, where

$$K_n = -(R + B' P_n B)^{-1} B' P_n A$$

and P_n is defined recursively by $P_N = Q$ and

$$P_{n-1} = Q + A'(P_n - P_n B(R + B' P_n B)^{-1} B' P_n)A.$$

The value function of this optimization is $V(x_0) = x_0' P_0 x_0$.

8 Infinite Horizon Linear Quadratic Regulator (LQR)

Consider the following optimization problem

$$\min \left\{ \sum_{n=0}^{\infty} x_n' Q x_n + u_n' R u_n : x_{n+1} = Ax_n + Bu_n \right\},$$

where $Q > 0$ and $R > 0$ are positive definite matrices. Note that this minimum may not be finite unless we impose additional restrictions.

In particular, suppose that (A, B) is stabilizable. Then the minimizer is given by $u_n = Kx_n$ (i.e., state-feedback with constant gain), where

$$K = -(R + B' P B)^{-1} B' P A$$

and $P > 0$ is the unique solution to the discrete time algebraic Riaccati equation (DARE)

$$P = Q + A'(P - PB(R + B'PB)^{-1}B'P)A.$$

The value function of this optimization is $V(x_0) = x_0'Px_0$. Furthermore, $A + BK$ is stable. There is an alternative characterization of this P as the solution to the following LMI:

$$\begin{aligned} &\max \ \mathrm{tr}(P) \\ &\text{s.t.} \ \ P \geq 0 \\ &\qquad \begin{bmatrix} A'PA + Q - P & B'PA \\ A'PB & R + B'PB \end{bmatrix} \geq 0. \end{aligned}$$