IEOR 290A – Lecture 38
Details for Single Utility Learning

1 Optimality-Conditions Feasibility Formulation

Recall the feasibility formulation when we can parameterize the utility function by $\phi(x, u; \beta)$, where this function is strictly concave in (x, u) for every fixed value of $\beta \in \Gamma$, with a gradient that is affine in β for every fixed value of (x, u):

$$\begin{aligned} \hat{\beta} = \arg\min_{\beta} \ & 0 \\ \text{s.t.} \ & -\nabla_x \phi(x_i^*, u_i; \beta) + \lambda_i A + \mu_i F = 0 \\ & \lambda_i^j \geq 0 \\ & \lambda_i^j = 0 \text{ if } A_j x_i^* + B_j u_i < c_j \\ & \beta \in \Gamma. \end{aligned}$$

Note that we will assume that $x \in \mathbb{R}^d$ and $u \in \mathbb{R}^q$.

2 Examples

This might seem like a restrictive formulation (in particular the requirement that the gradient is affine in β), but it can capture many useful situations. A few examples are described here.

2.1 Quadratic Utilities

Consider a quadratic utility given by

$$\phi(x, u; \beta) = -x'Qx + u'F'x + k'x,$$

where $Q \in \mathbb{R}^{d \times d} : Q \succeq 0$, $F \in \mathbb{R}^{q \times d}$ are matrices, and $k \in \mathbb{R}^d$ is a vector. Note that its gradient

$$\nabla_x \phi(x, u; \beta) = -2Qx + Fu + k.$$

is an affine function of the parameters Q, F, k.

The first thing to note is that this utility is equivalent to the following:

$$\tilde{\phi}(x, u; \tilde{\beta}) = -\begin{bmatrix} x \\ u \end{bmatrix}' \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}' \begin{bmatrix} x \\ u \end{bmatrix},$$

where $\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} \succeq 0$ is a block matrix that is appropriately sized, and $\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ is an appropriately sized block vector. The equivalence of this utility can be seen by noting that

$$\tilde{\phi}(x, u; \tilde{\beta}) = - \begin{bmatrix} x \\ u \end{bmatrix}' \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} + \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}' \begin{bmatrix} x \\ u \end{bmatrix} = -x'Q_{11}x - u'Q_{22}u + 2u'Q_{12}'x + k_1'x + k_2'u.$$

Since the utility maximizing agent optimizes over x for a fixed value of u, this means that the minimizer of this second problem will be equivalent to the first if $Q = Q_{11}$, $F = 2Q_{12}'$, and $k = k_1$.

The second thing to note is that there is a problem with the corresponding feasibility formulation

$$\begin{aligned} \hat{\beta} = \arg\inf_{\beta}\ & 0 \\ \text{s.t. }\ & 2Qx - Fu - k + \lambda_i A + \mu_i F = 0 \\ & \lambda_i^j \geq 0 \\ & \lambda_i^j = 0 \text{ if } A_j x_i^* + B_j u_i < c_j \\ & Q \succ 0. \end{aligned}$$

The following β is a feasible point of the above problem: $Q = 0$, $F = 0$, $k = 0$,$\lambda_i = 0$,and $\mu_i = 0$. This problem is a manifestation of the fact that there are an infinite number of utility functions that can lead to an observed set of decisions. To fix this problem, we must ensure that the formulation is properly normalized. One approach is to change the formulation to

$$\begin{aligned} \hat{\beta} = \arg\min_{\beta}\ & 0 \\ \text{s.t. }\ & 2Qx_i^* - Fu_i - k + \lambda_i A + \mu_i F = 0 \\ & \lambda_i^j \geq 0 \\ & \lambda_i^j = 0 \text{ if } A_j x_i^* + B_j u_i < c_j \\ & Q \succeq \mathbb{I}. \end{aligned}$$

2.2 Nonparametric Utilities

Instead of a parametric form of the utility, we can also define a nonparametric utility (essentially meaning an infinite number of parameters). For instance, we could have

$$\phi(x, u; \beta) = \sum_{i=0}^{\infty} k_i f_i(x, u),$$

where $f_i(x, u) : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}$ is a differentiable nonlinear function, and the β are the k_i parameters. In this case, the gradient is given by

$$\phi(x, u; \beta) = \sum_{i=0}^{\infty} k_i \nabla_x f_i(x, u),$$

which is affine in the β. Note that in general we will face a normalization issue, and so we would have to include an appropriate constraint in our feasibility problem to deal with this.

An example of the above is a finite polynomial expansion:

$$\phi(x,u;\beta) = k_1 x + k_2 x^2 + k_3 x u + k_4 x^2 u + k_5 x u^2,$$

in which case the feasibility problem with (one-potential) normalization is given by

$$\begin{aligned}
\hat{\beta} = \arg\min_{\beta}\ & 0 \\
\text{s.t.}\ & -k_1 - 2k_2 x_i^* - k_3 u_i - 2k_4 x_i^* u_i + k_5 u_i^* 2 + \lambda_i A + \mu_i F = 0 \\
& \lambda_i^j \geq 0 \\
& \lambda_i^j = 0 \text{ if } A_j x_i^* + B_j u_i < c_j \\
& k_2 \geq 1.
\end{aligned}$$

Here, we have chosen the normalization $k_2 \geq 1$. Note that we could have chosen other normalization constraints, such as $k_1 \geq 1$.

3 Suboptimal or Noisy Points

So far, we have assumed that the points (u_i, x_i^*) are measured without noise. Suppose instead that we measure $(u_i, x_i^* + \epsilon_i)$ where ϵ_i is some i.i.d. noise. (An alternative model is that the measured points (u_i, x_i) are suboptimal, meaning that they are close to the optimal values.) This introduces a new problem because now our optimality conditions will not be true. To overcome this difficulty, we define the new feasibility problem:

$$\begin{aligned}
\hat{\beta} = \arg\min_{\beta}\ & \sum_i r_{i,s}^2 + r_{i,c}^2 \\
\text{s.t.}\ & -\nabla_x \phi(x_i^*, u_i; \beta) + \lambda_i A + \mu_i F = r_{i,s} \\
& \lambda_i^j \geq 0 \\
& \lambda_i^j = r_{i,c} \text{ if } A_j x_i^* + B_j u_i < c_j \\
& \beta \in \Gamma.
\end{aligned}$$

The idea is that we allow for residuals in the equality constraints that would be identically zero for optimal points, to take into account that a measured point may be nonoptimal.