1 Utility Maximizing Agent

Recall the following abstract model: Suppose that an agent makes decisions by solving the following optimization problem:

$$x_i^* = \arg \max \{ J(x, u_i) \mid x \in \mathcal{X}(u_i) \}$$

where $u_i \in \mathbb{R}^d$ are inputs, $x_i^* \in \mathbb{R}^d$ are decisions, $J(x, u_i)$ is the utility function of the agent, and $\mathcal{X}(u_i)$ is a bounded set (that depends on $u_i$). In this model, we observe $(u_i, x_i^*)$ for $i = 1, \ldots, n$ and would like to infer the function $J(x, u_i)$.

To make this model more concrete, we will specify a specific instantiation of this problem. In particular, suppose that

- The constraint set is described by linear equality and inequality constraints:
  $$\mathcal{X}(u) = \{ x : Ax + B u_i \leq c, F x + G u_i = h \},$$
  
  where $(A, b)$ and $(F, h)$ are suitably defined matrices and vectors.

- Assume that we have a parametrization of the utility function, that is we have $\phi(x, u; \beta)$ and a bounded set $\Gamma$ such that there exists $\beta_0 \in \Gamma$ with $J(x, u) = \phi(x, u; \beta_0)$.

Though these two conditions make the problem more specific, we will still impose additional conditions on the model formulation to make the problem computationally tractable.

2 Key Technical Difficulty

Recall the feasibility problem formulation of the inverse decision-making problem for this single utility maximizing agent model:

$$\hat{\beta} = \arg \min_{\beta} 0$$

s.t. $x_i^* \in \arg \max_{x} \{ \phi(x, u_i; \beta) \mid Ax + B u_i \leq c, F x + G u_i = h \}$

$$\beta \in \Gamma.$$ 

This feasibility problem is difficult to solve because it has an atypical constraint: The constraint that $x_i^*$ be the minimizer to some optimization problem cannot be directly handled by nonlinear programming techniques. There are two reasons that this constraint presents challenges:
1. Depending on the value of $\beta$ there may be zero, one, or multiple maximizers. This means that in general we must treat the function 

$$P(u_i, \beta) = \arg \max_x \{\phi(x, u_i; \beta) \mid Ax + Bu_i \leq c, Fx + Gu_i = h\}$$

as a multi-valued function.

2. The function $P(u_i, \beta)$ has a complex form, because it is defined as a set of maximizers. This means that in general we cannot even hope for continuity of $P(u_i, \beta)$ (cf. the Berge Maximum Theorem, which says that for continuous $\phi$ we can only expect upper-hemicontinuity of $P(u_i, \beta)$), much less differentiability.

3. Tractable Formulation

Since $P(u_i, \beta)$ is a multi-valued function, we can make the problem more tractable by imposing additional conditions on our model so that instead $P(u_i, \beta)$ is a single-valued (and hence continuous by the Berge Maximum Theorem) function. In particular, suppose that for all fixed values of $\beta \in \Gamma$ the function $\phi(x, u_i; \beta)$ is strictly concave in $(x, u_i)$. Then the corresponding optimization problem has a single maximizer, and so this additional condition fixes our first difficulty.

The second difficulty regarding the complex form of $P(u_i, \beta)$ still remains. However, since our constraints are linear, we have linearity constraint qualification, and so the unique maximizer $x_i^* = P(u_i, \beta)$ satisfies the KKT conditions: There exist row-vectors $\lambda_i$ and $\mu_i$ such that

$$-\nabla_x \phi(x_i^*, u_i; \beta) + \lambda_i A + \mu_i F = 0$$

$$Ax_i^* + Bu_i \leq c$$

$$Fx_i^* + Gu_i = h$$

$$\lambda_i^j \geq 0$$

$$\lambda_i^j = 0 \text{ if } A_j x_i^* + B_j u_i < c_j,$$

where $A_j, B_j, c_j$ denote the $j$-th row of $A, B, c$ respectively. As a result, we can now pose our feasibility problem as

$$\hat{\beta} = \arg \min_{\beta} 0$$

s.t.  

$$-\nabla_x \phi(x_i^*, u_i; \beta) + \lambda_i A + \mu_i F = 0$$

$$\lambda_i^j \geq 0$$

$$\lambda_i^j = 0 \text{ if } A_j x_i^* + B_j u_i < c_j$$

$$\beta \in \Gamma.$$

Note that because $(u_i, x_i^*)$ are measured, they are constant in our feasibility formulation and in the KKT conditions. Therefore, the conditional statement “if $A_j x_i^* + B_j u_i < c_j$”
is computed before we solve the feasibility problem. In other words, we decide to either include or exclude the constraint $\lambda^j_i = 0$ in our feasibility problem based on a precomputed conditional.

This problem can still be difficult to solve, because this reformulated problem may not be convex. Consider the constraint

$$-\nabla x \phi(x^*_i, u_i; \beta) + \lambda_i A + \mu_i F = 0,$$

and note that it is an equality constraint. However, a standard result is that an equality constraint $Q(\beta)$ is convex if and only $Q$ is an affine function (meaning that it can be written as $Q = M \beta + k$ where $M$ is a matrix and $k$ is a constant vector). As a result, our feasibility problem to estimate the parameters $\beta$ of our utility function is convex if and only if $Q(\beta) = -\nabla x \phi(x^*_i, u_i; \beta)$ is an affine function. Stated in another way, our formulation is convex if and only if the gradient of $\phi$ with respect to $x$ is affine in $\beta$ when the gradient is evaluated at $x^*_i$ and $u_i$. 