IEOR 290A – Lecture 26
Selected Variational Analysis

1 Extended-Real-Valued Functions

A common formulation of a finite-dimensional optimization problem is

$$\begin{aligned} \min\ & f(x) \\ \text{s.t. } & g_i(x) \leq 0, \forall i = 1, \dots, I_1 \\ & h_i(x) = 0, \forall i = 1, \dots, I_2 \\ & x \in \mathcal{X} \subseteq \mathbb{R}^p \end{aligned}$$

where $f(x), g_i(x), h_i(x)$ are functions that have a domain that is a subset of $\mathbb{R}^p$, and $f(x)$ is a function with domain in $\mathbb{R}$. It turns out that for certain applications, it can be useful to redefine this optimization using extended-real-valued functions.

The extended-real-valued line is defined as $\overline{\mathbb{R}} = [-\infty, \infty]$ (compare this to the real-valued line $\mathbb{R} = (-\infty, \infty)$). The difference between these two lines is that extended-real-valued line specifically includes the values $-\infty$ and ∞, whereas these are not numbers in the real-valued line.

The reason that this concept is useful is that it can be used to reformulate the above optimization problem. In particular, suppose that we define an extended-real-valued function $\tilde{f}$ as follows

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } g_i(x) \leq 0, \forall i = 1, \dots, I_1; h_i(x) = 0, \forall i = 1, \dots, I_2; x \in \mathcal{X} \subseteq \mathbb{R}^p \\ \infty, & \text{otherwise} \end{cases}$$

We can hence formulate the above optimization problem as the following unconstrained optimization

$$\min \tilde{f}(x).$$

It is worth emphasizing this point: One benefit of formulating optimization problems using extended-real-valued functions is that it allows us to place the constraints and objective on equal footing.

2 Epigraph

An important concept in variational analysis is that of the epigraph. In particular, suppose we have an optimization problem

$$\min f(x),$$

where $f : \mathbb{R}^p \to \overline{\mathbb{R}}$ is an extended-real-valued function. We define the *epigraph* of f to be the set

$$\text{epi } f = \{(x, \alpha) \in \mathbb{R}^p \times \mathbb{R} \mid \alpha \geq f(x)\}.$$

Note that the epigraph of f is a subset of $\mathbb{R}^p \times \mathbb{R}$ (which does not include the extended-real-valued line).

3 Lower Semicontinuity

We define the *lower limit* of a function $f : \mathbb{R}^p \to \overline{\mathbb{R}}$ at $\overline{x}$ to be the value in $\overline{\mathbb{R}}$ defined by

$$\liminf_{x \to \overline{x}} f(x) = \lim_{\delta \searrow 0} \left[\inf_{x \in \mathcal{B}(\overline{x}, \delta)} f(x) \right] = \sup_{\delta > 0} \left[\inf_{x \in \mathcal{B}(\overline{x}, \delta)} f(x) \right],$$

where $\mathcal{B}(\overline{x}, \delta)$ is a ball centered at $\overline{x}$ with radius δ. Similarly, we define the *upper limit* of f at $\overline{x}$ as

$$\limsup_{x \to \overline{x}} f(x) = \lim_{\delta \searrow 0} \left[\sup_{x \in \mathcal{B}(\overline{x}, \delta)} f(x) \right] = \inf_{\delta > 0} \left[\sup_{x \in \mathcal{B}(\overline{x}, \delta)} f(x) \right].$$

We say that the function $f : \mathbb{R}^p \to \overline{\mathbb{R}}$ is lower semicontinuous (lsc) at $\overline{x}$ if

$$\liminf_{x \to \overline{x}} f(x) \geq f(\overline{x}), \text{or equivalently } \liminf_{x \to \overline{x}} f(x) = f(\overline{x}).$$

Furthermore, this function is lower semicontinuous on $\mathbb{R}^p$ if the above condition holds for every $\overline{x} \in \mathbb{R}^p$. There are some useful characterizations of lower semicontinuity:

- the epigraph set epi f is closed in $\mathbb{R}^p \times \mathbb{R}$;
- the level sets of type $\text{lev}_{\leq a} f$ are all closed in $\mathbb{R}^p$.

One reason that lower semicontinuity is important is that if f is lsc, level-bounded (meaning the level sets $\text{lev}_{\leq a} f$ are bounded), and proper (meaning that the preimage of every compact set is compact), then the value $\inf f$ is finite and the set $\arg\min f$ is nonempty and compact. This means that we can replace $\inf f$ by $\min f$ in this case.

4 Further Details

More details about these concepts can be found in the book *Variational Analysis* by Rockafellar and Wets, from which the above material is found.