IEOR 290A – Lecture 20
Linear MPC

1 Optimization Formulation

The issue with using a constant state-feedback controller $u_n = Kx_n$ is that the set Ω for which this controller is admissible can be quite small. In this sense, using this controller is conservative, and the relevant question is whether we can design a nonlinear controller $u_n = \ell(x_n)$ for the LTI system such that constraints on states $x_n \in \mathcal{X}$ and inputs $u_n \in \mathcal{U}$ are satisfied. Model predictive control (MPC), which is also known as finite horizon control (FHC), is a control technique that can provide exactly this.

One formulation for linear MPC is

$$\begin{aligned} V_n(x_n) = \min & \psi_n(\overline{x}_n, \ldots, \overline{x}_{n+N}, \check{u}_n, \ldots, \check{u}_{n+N-1}) \\ \text{s.t. } & \overline{x}_{k+1} = A\overline{x}_k + B\check{u}_k, \forall k = n, \ldots, n+N-1 \\ & \overline{x}_n = x_n \\ & \overline{x}_k \in \mathcal{X}, \forall k = n+1, \ldots, n+N-1 \\ & \overline{u}_k \in \mathcal{U}, \forall k = n, \ldots, n+N-1 \\ & \overline{x}_{n+N} \in \Omega \end{aligned}$$

where $\mathcal{X}, \mathcal{U}, \Omega$ are polytopes, and $N > 0$ is the horizon. Note that we do not constrain the initial condition x_n, and this reduces the conservativeness of the MPC formulation. We will refer to the set Ω as a terminal set. The interpretation of the function ψ_n is a cost function on the states and inputs of the system.

2 Recursive Properties

Suppose that (A, B) is stabilizable and K is a matrix such that $(A + BK)$ is stable. For this given system and feedback controller, suppose we have a maximal output invariant set Ω (meaning that this set has constraint satisfaction $\Omega \subseteq \{x : x \in \mathcal{X}, Kx \in \mathcal{U}\}$ and control invariance $(A + BK)\Omega \subseteq \Omega$).

Next, note that conceptually our decision variables are $\check{u}_k$ since the $\overline{x}_k$ are then uniquely determined by the initial condition and equality constraints. As a result, we will talk about solutions only in terms of the input variables $\check{u}_k$. In particular, if $\mathcal{M}_n = \{\check{u}_n, \ldots, \check{u}_{n+N-1}\}$ is feasible for the optimization defining linear MPC with an initial condition $x)_n$, then the system that applies the control value $u_n = \check{u}_n[\mathcal{M}_n]$ results in:

- Recursive Feasibility: there exists feasible $\mathcal{M}_{n+1}$ for x_{n+1};
- Recursive Constraint Satisfaction: $x_{n+1} \in \mathcal{X}$.

We will give a sketch of the proof for these two results.

- Choose $\mathcal{M}_{n+1} = \{\check{u}_{n+1}[\mathcal{M}_n], \dots, \check{u}_{n+N-1}[\mathcal{M}_n], K\overline{x}_{n+N}[\mathcal{M}_n]\}$. Since $\mathcal{M}_n$ is such that $\overline{x}_{n+N}[\mathcal{M}_n] \in \Omega$, then $K\overline{x}_{n+N}[\mathcal{M}_n] \in \mathcal{U}$ and $\overline{x}_{n+N+1}[\mathcal{M}_{n+1}] \in \Omega$. Also,

$$\check{u}_{n+1}[\mathcal{M}_n], \dots, \check{u}_{n+N-1}[\mathcal{M}_n] \in \mathcal{U},$$

  since $\mathcal{M}_n$ is feasible. Lastly $\overline{x}_{n+1+k}[\mathcal{M}_{n+1}] = \overline{x}_{n+1+k}[\mathcal{M}_{n+1}]$ for all $k = 1, \dots, N-1$, and so state constraint satisfaction holds since $\mathcal{M}_n$ is feasible and $\Omega \subseteq \mathcal{X}$.

- Since $x_{n+1} = \overline{x}_{n+1}[\mathcal{M}_n] \in \mathcal{X}$, the result follows.

There is an immediate corollary of this theorem: If there exists feasible $\mathcal{M}_0$ for x_0, then the system is (a) Lyapunov stable, (b) satisfies all state and input constraints for all time, (c) feasible for all time.

Lastly, define $\mathcal{X}_F = \{x_n |$ there exists feasible $\mathcal{M}_n\}$. It must be that $\Omega \subseteq \mathcal{X}_F$, and in general $\mathcal{X}_F$ will be larger. There is in fact a tradeoff between the values of N and the size of the set $\mathcal{X}_F$. Feasibility requires being able to steer the system into Ω at time N. The larger N is, the more initial conditions can be steered to Ω and so $\mathcal{X}_F$ will be larger. However, having a larger N means there are more variables and constraints in our optimization problem, and so we will require more computation for larger values of N.

3 Asymptotic Stability

There are two interesting results that act as lemmas. First, if ψ_n is continuous in $\overline{x}, \check{u}$, then $V_n(x_n)$ is continuous on $\text{int}(\mathcal{X}_F)$ by the Berge Maximum Theorem. Furthermore, if ψ_n is strictly convex, then $V_n(x_n)$ is convex and $u_n(x_n) = \check{u}_n[\mathcal{M}_n]$ is continuous in x_n, by the strictly convex variant of the Berge Maximum Theorem.

Now suppose we specialize the cost function to

$$\psi_n := \overline{x}'_{n+N} P \overline{x}_{n+N} + \sum_{k=n}^{n+N-1} \overline{x}'_k Q \overline{x}_k + \check{u}'_k R \check{u}_k,$$

where $Q > 0$, $R > 0$, and $P > 0$ such that P solves

$$(A + BK)' P (A + BK) - P = -(Q + K'RK).$$

Recall that if K and P are the respective matrices taken from the solution of the infinite horizon LQR problem, then this equality condition holds. Also note that this cost function is strictly convex.

It turns out that the LTI system using the control law provided by linear MPC is asymptotically stable, and the proof relies on a similar trick as was used to show recursive feasibility – namely utilizing the special properties related to $\overline{x}_{n+N}$ in the linear MPC formulation. We provide a sketch of the proof here: Note that since $\mathcal{M}_{n+1}$ as defined earlier is feasible, we must have that $V(x_{n+1}) \leq V(x_{n+1})[\mathcal{M}_{n+1}]$ (i.e., the optimal value is less than or equal to the value for any given feasible point). Next, observe that some algebra gives

$$\begin{aligned} V(x_{n+1})[\mathcal{M}_{n+1}] - V(x_n) &= \overline{x}'_{n+N+1} P \overline{x}_{n+N+1} + \overline{x}'_{n+N} Q \overline{x}_{n+N} \\ &\qquad - \overline{x}'_{n+N} P \overline{x}_{n+N} - \overline{x}'_n Q \overline{x}_n + \check{u}'_{n+N} R \check{u}_{n+N} - \check{u}'_n R \check{u}_n \\ &= \overline{x}'_{n+N}[(A+BK)' P (A+BK) - P + Q + K' R K] \overline{x}_{n+N} \\ &\qquad - \overline{x}'_n Q \overline{x}_n - \check{u}'_n R \check{u}_n \\ &\leq -\overline{x}'_n Q \overline{x}_n - \check{u}'_n R \check{u}_n \end{aligned}$$

This is zero if and only if $x_n = 0$ and $u_n = 0$. Thus, the value function of the linear MPC problem is strictly decreasing except when the system is at the origin. Some additional work shows that $V(x_n)$ satisfies all of the conditions of a Lyapunov function that shows asymptotic stability.