IEOR 290A – Lecture 18
Lyapunov Theory

1 Lyapunov Stability

Consider an autonomous nonlinear dynamical system in discrete time

$$x_{n+1} = f(x_n), \quad x_0 = \xi$$

with an equilibrium point at $x^* = 0$. Some important types of stability that we have covered are:

- A system is *Lyapunov stable* if given $M_2 > 0$ there exists $M_1 > 0$ such that $\|x_0\| \leq M_1$ implies that $\|x_n\| \leq M_2$ for all $n \geq 0$.
- A system is *locally asymptotically stable* (LAS) if (a) it is Lyapunov stable, and (b) there exists $M_3 > 0$ such that $\|x_n\| \to 0$ whenever $\|x_0\| \leq M_3$. A system is *globally asymptotically stable* (GAS) if $M_3 = \infty$.
- A system is *exponentially stable* if (a) it is asymptotically stable, and (b) there exists $M_3 > 0$ and $\alpha, \beta > 0$ such that $\|x_n\| \leq \alpha\|x_0\| \exp(-\beta n)$ whenever $\|x_0\| \leq M_3$. A system is *globally exponentially stable* if $M_3 = \infty$.

2 Lyapunov Function

Our tests for stability have been focused on discrete time LTI systems, and so it is natural to ask how to show stability for a nonlinear system in discrete time. To do so, we must first give some abstract definitions

- A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is type-$\mathcal{K}$ if it is continuous, strictly increasing, and $\gamma(0) = 0$.
- A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathcal{R}_+$ is type-$\mathcal{KL}$ if for each fixed $t \geq 0$, the function $\beta(\cdot, t)$ is type-$\mathcal{K}$, and for each fixed $s \geq 0$ the function $\beta(s, \cdot)$ is decreasing and $\beta(s, t) \to 0$ as $t \to \infty$.

With these definitions, we can now define a time-varying function that will indirectly allow us show that a system is stable. The function $V_n : \mathcal{X} \to \mathbb{R}$ is a Lyapunov function for a discrete time system if the following conditions hold:

1. $V_n(0) = 0$ and $V_n(x) > 0$ for all $x \neq 0$;
2. $\alpha_1(\|x\|) \leq V_n(x) \leq \alpha_2(\|x\|)$, where α_1, α_2 are type-$\mathcal{K}$ functions;

3. $x \in \text{int}(\mathcal{X})$, that is x is in the interior of $\mathcal{X}$;

4. $V_n(x_{n+1}) - V_n(x_n) \leq 0$;

The intuition of the first three conditions is that the Lyapunov function V_n is like an energy function that is zero at the equilibrium $x^* = 0$ and increases in value as it gets farther from the equilibrium. The last condition says that the value of the Lyapunov function evaluated at the current state of the system is non-increasing, and the intuition is that the energy of the system does not increase.

3 Lyapunov Theorems

These definitions lead to several theorems. In particular, we have that if an autonomous nonlinear discrete time system has:

- a Lyapunov function (as defined above with $V_n(x_n + 1) - V_n(x_n) \leq 0$), then then system is Lyapunov stable;
- a Lyapunov function such that $V_n(x_{n+1}) - V_n(x_n) < 0$ (that is strictly decreasing) for $x_n \neq 0$, then the system is LAS;
- a Lyapunov function such that $V_n(x_{n+1}) - V_n(x_n) < 0$ (that is strictly decreasing) for $x_n \neq 0$ and the domain of the Lyapunov function is $\mathcal{X} = \mathbb{R}^p$, then the system is GAS.
- a Lyapunov function such that $V_n(x_{n+1}) - V_n(x_n) < -\alpha V_n(x_n)$ for $x_n \neq 0$, V_n is such that $\alpha_1(\|x\|) = \kappa_1 \cdot \|x\|$ and $\alpha_2(\|x\|) = \kappa_2 \cdot \|x\|$, for some fixed $\alpha, \kappa_1, \kappa_1 > 0$, then the system is exponentially stable.
- a Lyapunov function such that $V_n(x_{n+1}) - V_n(x_n) < -\alpha V_n(x_n)$ for $x_n \neq 0$, V_n is such that $\alpha_1(\|x\|) = \kappa_1 \cdot \|x\|$ and $\alpha_2(\|x\|) = \kappa_2 \cdot \|x\|$, for some fixed $\alpha, \kappa_1, \kappa_1 > 0$, and the domain of the Lyapunov function is $\mathcal{X} = \mathbb{R}^p$, then the system is globally exponentially stable.

Note that we cannot use the above theorems to show that a system is unstable. This is in fact the biggest weakness of Lyapunov theory: There is no systematic way to compute a Lyapunov function for a system, unless the system is linear (or has polynomial dynamics).

4 Examples

We have already seen some examples of Lyapunov functions, specifically for LTI systems. Here, we recall the past examples and give a new example:

- Note that a discrete time system is (exponentially and asymptotically) stable if and only there exists $P > 0$ such that $A'PA - P < 0$, or equivalently given any $Q > 0$ there exists $P > 0$ such that $A'PA - P = -Q$. In this case, a Lyapunov function is $V(x) = x'Px$.

- Another example occurs in the infinite horizon LQR case. Consider the value function of the optimization

$$V(x_0) = \min \left\{ \sum_{n=0}^{\infty} x_n' Q x_n + u_n' R u_n : x_{n+1} = Ax_n + Bu_n; x_0 = \xi \right\},$$

where $Q > 0$ and $R > 0$ are positive definite matrices and (A, B) is stabilizable. The value function is equal to $V(x_0) = x_0' P x_0$ where $P > 0$ is the unique solution to the discrete time algebraic Riccati equation (DARE)

$$P = Q + A'(P - PB(R + B'PB)^{-1}B'P)A.$$

For the state-feedback of $u_n = Kx_n$ where $K = -(R + B'PB)^{-1}B'PA$, the value function of this optimization problem is a Lyapunov function for the closed-loop system $x_{n+1} = (A + BK)x_n$. In fact, a straightforward calculation gives

$$\begin{aligned}
&(A + BK)'P(A + BK) - P \\
&\quad = (A + BK)'P(A + BK) - P \\
&\quad = A'PA + K'B'PA + A'PBK + K'B'PBK - P \\
&\quad = A'PA + A'PBK + K'B'PA + K'(R + B'PB)K - K'RK - P \\
&\quad = A'PA + A'PBK + K'B'PA - K'B'PA - K'RK \\
&\quad = A'PA + A'PBK - P - K'RK
\end{aligned}$$

but the DARE can be rewritten as $P = Q + A'PA + A'PBK$, and so we have that

$$(A + BK)'P(A + BK) - P = A'PA + A'PBK - P - K'RK = -Q - K'RK < 0.$$

- As a final example, consider the following autonomous nonlinear system

$$x_{n+1} = x_n^2,$$

where $x_n, x_{n+1} \in \mathbb{R}$. If we choose $V(x) = x^2$, then we have that

$$V(x_{n+1}) - V(x_n) = x_n^4 - x_n^2 = x_n^2(x_n^2 - 1).$$

If $x_n^2 - 1 < 0$ (or equivalently $-1 < x_n < 1$), then $V(x_{n+1}) - V(x_n) < 0$. This choice of V satisfies the criterion for being a Lyapunov function for this particular system. Note that if $|x_n| > 1$ then the system is not stable in any sense.