IEOR 151 – Lecture 16
Capacitated Location Planning

1 Mathematical Model

We will consider extensions of different location planning models to the situation in which the facilities have a maximum capacity of demand that the facility is able to serve. The model for each model has similar elements: We represent the problem formulation using an undirected graph $G = (I, J, E)$, where the demand nodes are represented by a set of vertices $i \in I$, the possible locations of the facilities are given by another set of vertices $j \in J$, and edges $e_{i,j} \in E$ only exist between vertices from $i \in I$ to those in $j \in J$. Furthermore, we assign positive weights to the edges $d_{i,j} \geq 0$, which represents a distance between vertices i and j. Note that it is possible to have zero distance between a demand node and a possible facility location. We assign positive weights to the demand nodes h_i for $i \in I$, and this represents the amount of demand at a particular node. Moreover, each facility has a maximum capacity C of demand that can be served.

2 Capacitated p-Median Problem

In this model, we wish to place p facilities to minimize the (demand-weighted) average distance between a demand node and the location in which a facility was placed. In this extended model, there are constraints on the capacity of the facilities.

2.1 ILP Formulation

Because there are now capacity constraints for the facilities, we define a decision variable $Z_{i,j}$ that describes the amount of demand at demand node i that is serviced by a facility placed at j. We need to define another decision variable

$$X_j = \begin{cases} 1, & \text{if facility located at } j \in J \\ 0, & \text{otherwise} \end{cases}$$

that describes the locations at which a facility is placed. Given these decision variables, we can now formulate the capacitated p-median problem as the following ILP

$$\begin{aligned}
\min & \sum_{j\in J}\sum_{i\in I} d_{i,j}Z_{i,j} \\
\text{s.t.} & \sum_{j\in J} Z_{i,j} = h_i, \forall i \in I \\
& Z_{i,j} \geq 0, \forall i \in I, j \in J \\
& Z_{i,j} \leq h_i X_j, \forall i \in I, j \in J \\
& \sum_{j\in J} X_j = p \\
& \sum_{i\in I} Z_{i,j} \leq C, \forall j \in J \\
& X_j \in \{0,1\}, \forall j \in J.
\end{aligned}$$

Each of these terms has associated intuition:

- The objective $\sum_{j\in J}\sum_{i\in I} d_{i,j}Z_{i,j}$ is stating that we wish to minimize the demand-weighted (i.e., weighted by h_i) distance $d_{i,j}Z_{i,j}$ summed over all facilities and demand nodes.
- The first constraint $\sum_{j\in J} Z_{i,j} = h_i$ implies that a demand node i must have all of its demand met.
- The constraint $Z_{i,j} \geq 0$ says that the amount of demand served must be non-negative, and $Z_{i,j} \leq h_i X_j$ says that demand node i can be serviced by a facility at j only if there is a facility at j, because if $X_j = 0$ then we must have that $Z_{i,j} = 0$.
- The constraint $\sum_{j\in J} X_j = p$ means that we must place exactly p facilities.
- The constraint $\sum_{i\in I} Z_{i,j} \leq C$ says that a facility at j can service at most C amount of demand.
- Lastly, the constraint that $X_j \in \{0,1\}$ forces this decision variables to be binary.

Note that we do NOT need a constraint of the form $\sum_{i\in I} Z_{i,j} \leq CX_j$ because if $X_j = 0$ then we already have that $Z_{i,j} = 0$.

3 Capacitated Vertex p-Center Problem

In this problem, we would like to place p facilities to minimize the maximum distance between any demand node and its servicing facility. In this extended model, there are constraints on the capacity of the facilities.

3.1 ILP Formulation

Because there are now capacity constraints for the facilities, we define a decision variable $Z_{i,j}$ that describes the amount of demand at demand node i that is serviced by a facility placed at j. We define a decision variable

$$Y_{i,j} = \begin{cases} 1, & \text{if demand node } i \in I \text{ assigned to facility located at } j \in J \\ 0, & \text{otherwise} \end{cases}$$

that describes which demand nodes are serviced by which facility location. We need to define another decision variable

$$X_j = \begin{cases} 1, & \text{if facility located at } j \in J \\ 0, & \text{otherwise} \end{cases}$$

that describes the locations at which a facility is placed. In contrast to the (capacitated) p-median problem, we must define an additional decision variable Q that represents the maximum distance between any demand node and its servicing facilities. Given these decision variables, we can now formulate the capacitated vertex p-center problem as the following ILP

$$\begin{aligned}
\min\ & Q \\
\text{s.t. } & \sum_{j \in J} Z_{i,j} = h_i, \forall i \in I \\
& Z_{i,j} \geq 0, \forall i \in I, j \in J \\
& Z_{i,j} \leq h_i X_j, \forall i \in I, j \in J \\
& Z_{i,j} \leq h_i Y_{i,j}, \forall i \in I, j \in J \\
& Y_{i,j} \leq X_j, \forall i \in I, j \in J \\
& \sum_{j \in J} X_j = p \\
& \sum_{i \in I} Z_{i,j} \leq C, \forall j \in J \\
& Q \geq d_{i,j} Y_{i,j}, \forall i \in I, j \in J \\
& X_j \in \{0, 1\}, \forall j \in J \\
& Y_{i,j} \in \{0, 1\}, \forall i \in I, j \in J.
\end{aligned}$$

Each of these terms has associated intuition:

- The objective Q is stating that we wish to minimize the maximum distance between any demand node and its servicing facility.
- The first constraint $\sum_{j \in J} Z_{i,j} = h_i$ implies that a demand node i must have all of its demand met.

- The constraint $Z_{i,j} \geq 0$ says that the amount of demand served must be non-negative, and $Z_{i,j} \leq h_i X_j$ says that demand node i can be serviced by a facility at j only if there is a facility at j, because if $X_j = 0$ then we must have that $Z_{i,j} = 0$.
- The constraint $Y_{i,j} \leq X_j$ means that i can be served by j only if a facility is placed at j, and $Z_{i,j} \leq h_i Y_{i,j}$ means that if $Z_{i,j} > 0$ then $Y_{i,j}$ is forced to be 1 since it is binary.
- The constraint $\sum_{j \in J} X_j = p$ means that we must place exactly p facilities.
- The constraint $\sum_{i \in I} Z_{i,j} \leq C$ says that a facility at j can service at most C amount of demand.
- The constraint $Q \geq d_{i,j} Y_{i,j}$ means that Q must be greater than the distance between the i-th demand node and its servicing facility. And because we have this constraint for each of the $i \in I$ demand nodes and $j \in J$ potential facility locations, this means that Q must be greater than the distance of any demand node and its servicing facility.
- The constraints that $X_j \in \{0, 1\}$ and $Y_{i,j} \in \{0, 1\}$ force these decision variables to be binary.

Note that there is an important subtlety here: The constraint $Z_{i,j} \leq h_i Y_{i,j}$ means that if $Z_{i,j} = 0$ then $Y_{i,j}$ can be either 0 or 1. This may seem undesirable because if $Z_{i,j} = 0$ then this means that j is not servicing i, and so we would want $Y_{i,j} = 0$. However, this condition occurs implicitly because of the constraint $Q \geq d_{i,j} Y_{i,j}$. If $Z_{i,j} = 0$, then choosing $Y_{i,j} = 0$ is preferable since this would possibly allow Q to take on a smaller value.

4 Capacitated Set Covering Problem

In this problem, a facility can serve demand nodes that are within a given coverage distance D^c from the facility. The problem is to place the minimum number of facilities so as to ensure that all demand nodes can be served. In this extended model, there are constraints on the capacity of the facilities. We define the indicator variable

$$a_{i,j} = \begin{cases} 1, & \text{if } d_{i,j} \leq D^c \\ 0, & \text{otherwise} \end{cases}$$

that describes if demand node at $i \in I$ can be covered by a facility at $j \in J$ if the coverage distance is D^c.

4.1 ILP Formulation

Because there are now capacity constraints for the facilities, we define a decision variable $Z_{i,j}$ that describes the amount of demand at demand node i that is serviced by a facility placed at j. We define a decision variable

$$X_j = \begin{cases} 1, & \text{if facility located at } j \in J \\ 0, & \text{otherwise} \end{cases}$$

that describes the locations at which a facility is placed. Given these decision variables, we can now formulate the capacitated set covering problem as the following ILP

$$\begin{aligned}
\min & \sum_{j \in J} X_j \\
\text{s.t.} & \sum_{j \in J} Z_{i,j} = h_i, \forall i \in I \\
& Z_{i,j} \geq 0, \forall i \in I, j \in J \\
& Z_{i,j} \leq h_i \cdot a_{i,j} \cdot X_j, \forall i \in I, j \in J \\
& \sum_{i \in I} Z_{i,j} \leq C, \forall j \in J \\
& X_j \in \{0, 1\}, \forall j \in J.
\end{aligned}$$

Each of these terms has associated intuition:

- The objective $\sum_{j \in J} X_j$ is stating that we wish to minimize the number of placed facilities.
- The first constraint $\sum_{j \in J} Z_{i,j} = h_i$ implies that a demand node i must have all of its demand met.
- The constraint $Z_{i,j} \geq 0$ says that the amount of demand served must be non-negative, and $Z_{i,j} \leq h_i \cdot a_{i,j} \cdot X_j$ says that demand node i can be serviced by a facility at j only if there is a facility at j and if the distance between i and j is less than D^c. This is because if $X_j = 0$ or $a_{i,j} = 0$ then we must have that $Z_{i,j} = 0$.
- The constraint $\sum_{i \in I} Z_{i,j} \leq C$ says that a facility at j can service at most C amount of demand.
- The constraint $X_j \in \{0, 1\}$ forces these decision variables to be binary.