The Euler Tour and Chinese Postman Problem

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IEOR 151

Related Service Problems

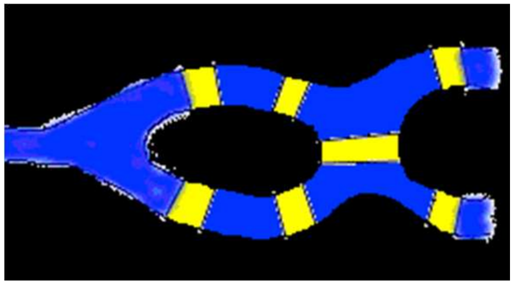
- Node routing problems:
  - Meal delivery, inter-library loans, school-bus routing
- Arc routing problems:
  - Waste collection, snow plowing, postal delivery route design

Arc Routing Problems

- We want to traverse or travel along each street of a road system at least once
- Question: how to do this with minimal time/cost while maintain a certain service level?
- People have thought about this problem looong time ago…

The Seven Bridges of Konigsberg

- In Konigsberg, Germany, a river ran through the city such that in its center was an island, and after passing the island, the river broke into two parts. Seven bridges were built so that the people of the city could get from one part to another. A crude map of the center of Konigsberg might look like this:



- The people wondered whether or not one could walk around the city in a way that would involve crossing each bridge exactly once.

Euler and Graph Theory

- This long-standing problem was solved in 1735 in an ingenious way by the Swiss mathematician Leonhard Euler (1707-1782).
- His solution, and his generalization of the problem to an arbitrary number of islands and bridges, gave rise to a very important branch of mathematics called Graph Theory.

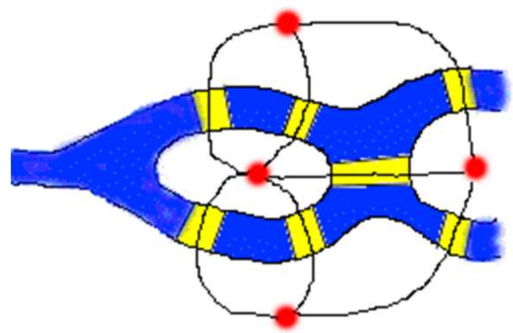
The Chinese Postman Problem

- A similar problem is called Chinese Postman Problem (after the Chinese mathematician, Kwan Mei-Ko, who discovered it in early 1960's).
- It is the problem that the Chinese Postman faces: he wishes to travel along every road in a city in order to deliver letters, with the least possible distance.
- The problem is how to find a shortest closed walk of the graph in which each edge is traversed <u>at least once</u>, rather than exactly once.

Graph Theory

- A graph consists of a non-empty set of points (vertices) and a set of lines (edges) connecting the vertices.
- The number of edges linked to a vertex is called the <u>degree</u> of that vertex.
- A walk, which starts at a vertex, traces each edge <u>exactly once</u> and ends at the starting vertex, is called an <u>Euler Trail</u>.
  - If it ends at some other vertex, it is called an <u>open Euler trail</u>. The Königsberg Bridges problem was an attempt to find an open Euler trail.

The Seven Bridges of Konigsberg



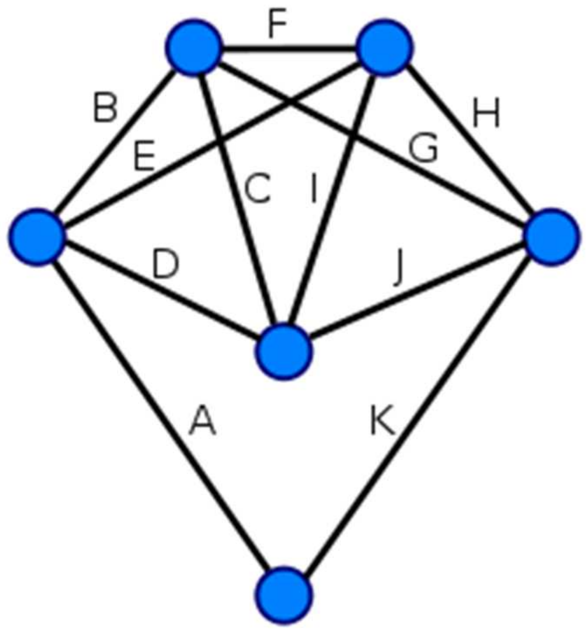
- Definition: A vertex is called odd if it has an odd number of arcs leading to it, other wise it is called even.

- **Theorem:**

An open Euler trail is possible if and only if there are exactly two vertices of odd degree.

An Euler trail is possible if and only if every vertex is of even degree.

Euler Trial



- Every vertex of this graph has an even degree, therefore this is a Euler graph. Following the edges in alphabetical order gives a Euler trail.

Constructing Euler Trails

- Hierholzer's 1873 paper:
  - Choose any starting vertex *v*, and follow a trail of edges from that vertex until returning to *v*. The tour formed in this way is a closed tour, but may not cover all the vertices and edges of the initial graph.
  - As long as there exists a vertex *v* that belongs to the current tour but that has adjacent edges not part of the tour, start another trail from *v*, following unused edges until returning to *v*, and join the tour formed in this way to the previous tour.

Constructing Euler trails

- Question: is it possible to get stuck at any vertex other than v?

- It is not possible to get stuck at any vertex other than v, because the even degree of all vertices ensures that, when the trail enters another vertex w there must be an unused edge leaving w.

Chinese Postman Problem

- For the practical situation, the problems like delivery of mail or newspaper, trash pick-up, and snow removal can be modeled by Chinese Postman Problem.
- The problem can be solved in polynomial time if all edges of the graph are undirected.

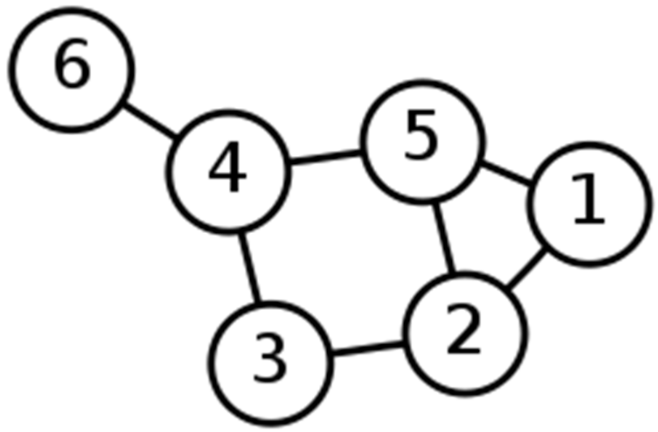
Solving CPP

- If a graph has an Euler trail, the solution is to choose the Euler trail.
- If the graph is not Eulerian, it must contain vertices of odd degree. By the handshaking lemma, there must be an even number of these vertices.

Handshaking Lemma

- The **handshaking lemma** is the statement that every finite undirected graph has an even number of vertices with odd degree.
- In a party of people some of whom shake hands, an even number of people must have shaken an odd number of other people's hands.

Handshaking Lemma



- In this graph, an even number of vertices (the four vertices numbered 2, 4, 5, and 6) have odd degrees. The sum of the degrees of the vertices is 2 + 3 + 2 + 3 + 3 + 1 = 14, twice the number of edges.

Solving CPP

- Let T be a subset of the vertex set of a graph. An edge set whose odd-degree vertices are the vertices in T is called a ***T*-join**.
- To solve the postman problem we first find a smallest T-join. We make the graph Eulerian by doubling of the T-join. The solution to the postman problem in the original graph is obtained by finding an Eulerian circuit for the new graph.

Example