PAPER NO. THP3.1

OPTIMAL BIDDING IN SEQUENTIAL AUCTIONS

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Abstract

When a bidder's strategy in one auction will affect his competitor's behavior in subsequent auctions, bidding in a sequence of auctions can be modeled fruitfully as a multistage control process in which the control is the bidder's strategy while the state characterizes the competitors' behavior. This paper presents such a model in which the state transition represents the competitors' reaction to the bidder's strategy. Dynamic programming is used to derive the infinite horizon optimal bidding strategy. It is shown that in steady state this optimal strategy generalizes a previous result for equilibrium bidding strategy in "one-shot" auctions.

1. Introduction

Much of the theory of competitive bidding and all of the early developments in that theory dealt with "one-shot" situations. By one shot situations, we mean bidding situations in which it is appropriate for the bidder to attempt to maximize his expected profit from the present auction or simultaneous group of auctions. Recently, a number of models for optimum bidding in sequential auctions have been developed [1,2,3,6, 7,8,13,14]. All of these deal with the internal effects within the bidding firm of winning or losing auctions. With the exception of a little known and specialized paper of Bauerjee and Ghosh [2], they all share the assumption that the competition will not react in later auctions to what the bidder has done in earlier auctions.* At times this must be a tenuous assumption in a field that is filled with literature suggesting how bidders should use information about the past behavior by competitors to determine their bids in present auctions. Therefore, we have built a model of bidding in sequential auctions in which a bidder's competitors are assumed to react to his previous bids.

One possible approach is to model the bidding process as a dynamic multiplayer noncooperative game. This, however, seems difficult. Our approach is less ambitious. We assume that bidders develop a behavioral model of how their competitors will react. This leads us to model the sequential bidding problem as a multistage control process in which the control is the bidder's strategy while the state characterizes the collective behavior of the competitors. In this model the state transition represents the competitors' reactions to the bidder's policy. Dynamic programming is used to derive an equation for the optimal infinite horizon bidding strategy. When this equation is solved for a generalization of a previous model of "one-shot" auctions [9], the formula for the optimal "one-shot" policy is modified by the inclusion of a term that depends upon the magnitude of competitive reaction, the time between auctions, and the discount rate.

* In addition, there are two works that deal with the problem of two bidders bidding sequentially for two items. [5,11]

2. A Dynamic Model of Bidding in Sequential Auctions

Consider a bidder who faces an infinite sequence of sealed bid auctions. In each auction, his expected profit from that auction depends upon the bidding policy of the rest of the industry and upon his own bidding policy. He would like to choose his own policy to maximize his expected profit from the immediate auction. However, he must keep in mind that his actions in the present auction will be observed by his competitors and will influence their behavior in future auctions. Therefore, it is in his interest to choose a bidding policy that will maximize his expected present value of profits in the present auction and all future auctions. It is this process that we propose to model.

Let $\underset{\sim}{P}(k)$ be an dimensional vector where n is the total number of bidders and $P_i(k)$, the i^{th} component of $\underset{\sim}{P}(k)$, is a scalar representing the bidding policy of the i^{th} bidder in the k^{th} auction. The expected reward of the i^{th} bidder in the k^{th} auction, denoted by $E_i(k)$, is assumed to be a fixed function of $\underset{\sim}{P}(k)$. Furthermore, this reward function is assumed to have the form

$$E_i(k) = E_i(Q_i(k)\ ,\ P_i(k))\ , \tag{1}$$

where $Q_i(k)$ is a scalar function of all the components of $\underset{\sim}{P}(k)$ excluding $P_i(k)$ such that if $P_j(k) = P$ for all $j \neq i$ then $Q_i(k) = P$. The reward function $E_i(\cdot)$ summarizes all of the internal effects on the bidder of winning or losing a particular auction, along with his likelihood of winning.

Under these assumptions the i^{th} bidder may view $Q_i(k)$ as a variable representing the aggregate bidding policy of the rest of the trade. Furthermore, the effect of his present bidding policy on the future behavior of the rest of the trade may be encoded in terms of the change in $Q_i(k)$. In particular, we shall assume that at any auction $k+1$

$$Q_i(k+1) = f_i(Q_i(k)\ ,\ P_i(k))\ . \qquad (2)$$

This implies that the aggregate policy of the trade, as viewed by the i^{th} bidder, in the $(k+1)^{th}$ auction depends on this aggregate policy and the policy of the i^{th} bidder in the k^{th} auction. Equation (2) may be viewed as a behavioral assumption on how the i^{th} bidder views the dynamics of the trade. However, it may also be regarded as an approximation to the update of $Q_i(k)$ that could result from a game theoretic approach.

The above assumptions enable the i^{th} bidder to view the process of bidding in sequential auction as a multistage control process (see [4]) where $P_i(k)$ is the control, $Q_i(k)$ is a state variable and Equation (2) is the state transition function. Bidder i's objective is to determine a control sequence that will maximize his present value of the rewards over an infinite horizon. Let D_i be the discount factor of the i^{th} bidder, i.e., $D_i = \exp(-r_i t)$ where t is the time between auctions and r_i is the continuous discount rate of the i^{th} bidder. Then bidder i's problem is to maximize $\sum_{k=0}^{\infty} D_i^k E_i(k)$ where $E_i(k)$ is as in (1).

This problem can be solved using dynamic programming. Let

$$V_i(Q_i,j) = \max_{\{P_i(k)\}_j^\infty} \sum_{k=j}^{\infty} D_i^{(k-j)} E_i(Q_i(k),P_i(k)) \qquad (3)$$

$$\text{Subject to:}\quad Q_i(k+1) = f_i(Q_i(k),P_i(k))\ ,$$

$$Q_i(j) = Q_i$$

One can easily show that for $D_i < 1$,

$$V_i(Q_i,j) = V_i(Q_i,j+1) = V_i(Q_i)\ . \qquad (4)$$

Thus the argument j can be suppressed, and by the "principle of optimality" we have

$$V_i(Q_i) = \max_P \{E_i(Q_i,P) + D_i V_i[f_i(Q_i,P)]\} \qquad (5)$$

For notational convenience we shall omit temporarily the subscripts i. Let $\hat{P}(Q)$ be the value of P that maximizes the right hand side of (5) for a given value of Q, then

$$V(Q) = E(Q,\hat{P}(Q)) + D\ V(f(Q,\hat{P}(Q)))\ . \qquad (6)$$

Assuming $E(Q,P)$ and $f(Q,P)$ are differentiable with respect to P, $P(Q)$ is a stationary point satisfying the necessary condition:

$$\left\{\frac{\partial E(Q,P)}{\partial P} + D\cdot \left.\frac{dV(\xi)}{d\xi}\right|_{\xi=f(Q,P)} \cdot \frac{\partial f(Q,P)}{\partial P}\right\}_{P=\hat{P}(Q)} = 0 \qquad (7)$$

for any given Q.

Substituting ξ for Q in (6) and differentiating with respect to ξ yields

$$\frac{dV(\xi)}{d\xi} = \left\{\frac{\partial E(\xi,\eta)}{\partial \xi} + D\left.\frac{dV(\mu)}{d\mu}\right|_{\mu=f(\xi,\eta)} \cdot \frac{\partial f(\xi,\eta)}{\partial \xi}\right\}_{\eta=\hat{P}(\xi)} + \left\{\frac{\partial E(\xi,\eta)}{\partial \eta} + D\left.\frac{dV(\mu)}{d\mu}\right|_{\mu=f(\xi,\eta)} \cdot \frac{\partial f(\xi,\eta)}{\partial \eta}\right\}_{\eta=\hat{P}(\xi)} \cdot \frac{d\hat{P}(\xi)}{d\xi}\ . \qquad (8)$$

The second part of (8) is zero by (7). Solving for $D[dV(\xi)/d\xi]$ at $\xi=f(Q,P(Q))$ and substituting it into the remainder of (8) gives

$$\frac{dV(\xi)}{d\xi} = \left\{\frac{\partial E(\xi,\eta)}{\partial \xi} - \frac{\partial E(\xi,\eta)}{\partial \eta}\cdot\frac{\partial f(\xi,\eta)/\partial \xi}{\partial f(\xi,\eta)/\partial \eta}\right\}_{\eta=\hat{P}(\xi)}\ . \qquad (9)$$

Substituting back into (7) and restoring the subscript i yields

$$\left\{\frac{\partial E_i(Q_i,\eta)}{\partial \eta} + D_i\frac{\partial f_i(Q_i,\eta)}{\partial \eta}\left[\frac{\partial E_i(\xi,\eta)}{\partial \xi} - \frac{\partial E_i(\xi,\eta)}{\partial \eta}\ \frac{\partial f_i(\xi,\eta)/\partial \xi}{\partial f_i(\xi,\eta)/\partial \eta}\right]\right\}_{\substack{\eta=\hat{P}_i(Q_i)\\ \xi=f_i(Q_i,\hat{P}_i(Q_i))}} = 0\ . \qquad (10)$$

Equation (10) is a necessary condition for the optimal strategy $\hat{P}_i(Q_i)$. According to this strategy, bidder i's optimal policy in any auction will be

$$P_i^*(k) = \hat{P}_i(Q_i(k)) \qquad . \qquad (11)$$

It should be noted that $\hat{P}_i(Q_i(k))$ is bidder i's optimal policy given his assumption that the trade's behavior is represented by $Q_i(k)$ and Eq. (2). This is true independently of whether bidder i's previous policies were optimal and of whether the trade has followed the assumed reaction function in the past.

3. Optimal Equilibrium Policy for Identical Bidders

To obtain more specific results we shall consider now the special case of identical bidders i.e., when $f_i(\cdot)$, $E_i(\cdot)$ and D_i are the same for all i. It is conceivable that there exist in this case an optimal equilibrium policy P^* such that, if all the bidders use this policy it is optimal for each of them to keep using it. This implies $\hat{P}_i(P^*) = P^*$ for all i. Furthermore, for the reaction function $f_i(\cdot)$ to be consistent dynamically with such behavior it has to satisfy

$$f_i(P^*,P^*) = P^*\ . \qquad (12)$$

Substituting these two conditions in Eq. (10) and eliminating the subscript i, results in the following equation for P^*.

$$\left\{\frac{\partial E(\xi,\eta)}{\partial \eta} + D\left[\frac{\partial E(\xi,\eta)}{\partial \xi}\cdot\frac{\partial f(\xi,\eta)}{\partial \eta} - \frac{\partial E(\xi,\eta)}{\partial \eta}\cdot\frac{\partial f(\xi,\eta)}{\partial \xi}\right]\right\}_{\xi=\eta=P^*} = 0\ . \qquad (13)$$

For the policy P^* to be a meaningful equilibrium policy, it should be a stable equilibrium in the sense that small deviations from this policy will create incentives that will drive the trade back to equilibrium. This stability condition imposes an additional restriction on the reaction function. In particular, $f(\cdot)$ has to be such that for sufficiently small ΔP

$$|f(P^*+\Delta P\ ,\ \hat{P}(P^*+\Delta P)) - P^*| \le \theta|\Delta P| \text{ for some } \theta < 1 \ . \quad (14)$$

For $\Delta P \to 0$ Eq. (14) becomes

$$\left|\left.\frac{\partial f(\xi,\eta)}{\partial \xi} + \frac{\partial f(\xi,\eta)}{\partial \eta} \cdot \frac{d\hat{P}(\xi)}{d\xi}\right|_{\xi=\eta=P^*}\right| < 1 \quad . \quad (15)$$

In view of the above discussion it is clearly desirable from a game theory point of view to choose a reaction function $f(\cdot)$ satisfying Eq. (12), (13), and (15). Using such a response function leads to an equilibrium policy which is consistent with the game theoretic approach in which all bidders are assumed to optimize their policy while considering that their competitors do the same.

4. Reaction Functions

In this section we shall discuss two specific forms for the reaction function $f(\cdot)$ in our model, and their implications.

The functional form we have chosen to consider initially is

$$f(Q_i,P_i) = Q_i - \alpha\left(\frac{1}{n} - p_i\right)(P_i - Q_i) \quad . \quad (16)$$

In this equation, Q_i and P_i are the respective policies of the trade and bidder i, n is the total number of bidders, α is a scalar parameter and p_i is the winning probability of bidder i. The probability p_i is in general a function of the number of bidders n and the policies Q_i and P_i.

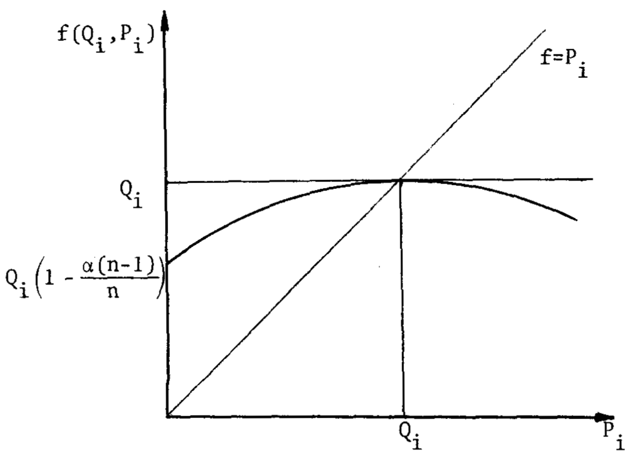

Figure 1: $f(Q_i,P_i) = Q_i - \alpha(\frac{1}{n} - p_i)(P_i - Q_i)$

The reaction function defined by (16) corresponds to an assumption that the trade reaction to a deviation from trade policy by bidder i will be proportional to the product of the extent of the deviation and the extent by which bidder i deviates from his "fair" share, $1/n$, of the market. The constant α is, of course, the constant of proportionality. We assume it to be positive. With this definition, in the case of identical bidders, an equilibrium policy implies $Q_i = P_i = P^*$ and $p_i = 1/n$. Thus (16) satisfies condition (12).

Figure 1 illustrates $f(Q_i, P_i)$ as a function of P_i for a given Q_i. Assuming that $P_i = 0$ implies $p_i = 1$, it begins with the value $f(Q_i,0) = Q_i(1-\alpha(n-1)/n)$. Then it rises to a maximum of Q_i at $P_i = Q_i$ and then declines.

This choice of reaction function has several important consequences. One can easily verify that for this function,

$$\left.\frac{\partial f(\xi,\eta)}{\partial \eta}\right|_{\xi=\eta=P^*} = 0 \quad (17)$$

and

$$\left.\frac{\partial f(\xi,\eta)}{\partial \xi}\right|_{\xi=\eta=P^*} = 1 \quad . \quad (18)$$

Thus, if we substitute (17) and (18) in (13) and assume $D < 1$ we obtain the equilibrium condition:

$$\left.\frac{\partial E(P^*,\eta)}{\partial \eta}\right|_{\eta=P^*} = 0 \quad . \quad (19)$$

This is the same condition as for the "one-shot" situation. Substituting (17) and (18) in the right hand side of (15) indicates, however, that P^* satisfying (19) is an unstable equilibrium. This instability has disconcerting implications with respect to the dynamics of the trade policy. If the initial trade policy, Q_i is above the static equilibrium value, P^*, that satisfies (19), then there is no problem. However, if Q_i is below P^*, then the equilibrium can never be reached. Any deviation by an optimizing bidder from the trade policy will only lead the members of the trade to lower their bids and thus move them further from the "equilibrium". In this range, the model is analagous to the "kinked demand curve" model of oligopoly theory in which competitors match price decreases but not price increases.

The second reaction function that we consider is

$$f(Q_i,P_i) = Q_i + \alpha p_i(P_i - Q_i) \quad (20)$$

where Q_i, P_i, α and p_i, are as in Eq. (16). This definition corresponds to an assumption that the trade reaction to a deviation from trade policy is proportional to the deviation and to the likelihood of the deviating bidder winning the auction, and it clearly satisfies (12). This reaction function considered as a function of P_i for given Q_i is illustrated in Figure 2. Again assuming $P_i = 0$ implies $p_i = 1$, it starts with $f(Q_i,0) = (1-\alpha)Q_i$. It then rises (falls for $\alpha < 0$) with a slope that starts at α and decreases to α/n by the time P_i has been increased to Q_i. At this point $f(Q_i,Q_i) = Q_i$. It continues to increase (decrease if $\alpha < 0$)

until it reaches an extremum and then approaches Q_i asymptotically for large P_i. In order to insure that $f(\cdot)$ is positive for every positive P_i we must restrict α to be less than unity. Furthermore, since in the identical bidders case, negative values of α can be shown to result in an unstable equilibrium, we shall restrict α to be positive.

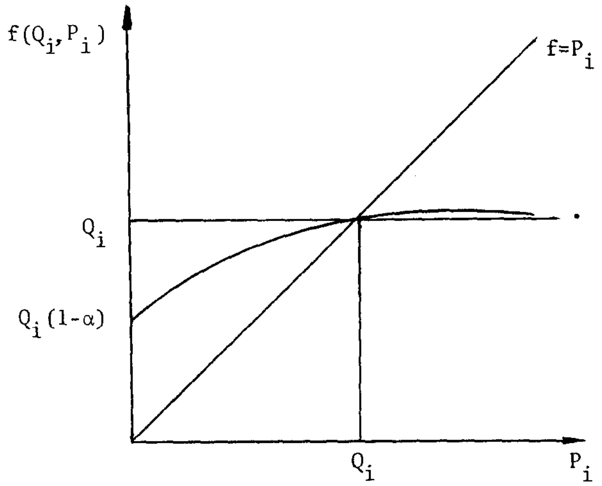

Figure 2: $f(Q_i, P_i) = Q_i + \alpha p_i(P_i - Q_i)$

Again in the case of identical bidders, an equilibrium policy implies $Q_i = P_i = P^*$ and $p_i = 1/n$. Thus for the reaction function (20)

$$\frac{\partial f(\xi,\eta)}{\partial \eta}\bigg|_{\xi=\eta=P^*} = \frac{\alpha}{n} \quad . \tag{21}$$

and

$$\frac{\partial f(\xi,\eta)}{\partial \xi}\bigg|_{\xi=\eta=P^*} = 1 - \frac{\alpha}{n} \tag{22}$$

Substituting (21) and (22) in (13) yields the equilibrium condition

$$\left\{\frac{\partial E(\xi,\eta)}{\partial \eta}\left(1 - D\left(1 - \frac{\alpha}{n}\right)\right) + \frac{\partial E(\xi,\eta)}{\partial \xi} D \frac{\alpha}{n}\right\}_{\xi=\eta=P^*} = 0 \cdot \tag{23}$$

In this case, the sequential auction equilibrium satisfying (23) is generally different from the one shot equilibrium mentioned earlier. Two exceptional situations in which (23) reduces to (19) and hence the two equilibriums are the same occur when $D=0$ or $\frac{\alpha}{n}=0$. In the first case the bidders disregard future payoffs and thus behave as in a one shot situation. In the second case the trade is insensitive to individual policies either because $\alpha=0$ or because the number of bidders, n, is large. Thus, an individual bidder should not worry about his impact on the trade and can behave as in a one shot auction.

Substituting (21) and (22) in (15) yields the stability condition for the equilibrium policy satisfying (23). This condition is

$$\frac{d\hat{P}(\xi)}{d\xi}\bigg|_{\xi=P^*} < 1 \quad . \tag{24}$$

Since $\hat{P}(P^*) = P^*$, Eq. (24) implies that P^* is a stable policy if near P^* the optimal policy of each bidder deviates from P^* less than his estimate of the trade policy. A rigorous proof that this condition is satisfied involves the specific form of the reward function. However, in general one can expect that if the trade bids very aggressively, bidder i cannot make money in this auction and thus his optimal strategy will be to lose the bid by bidding very unaggressively. On the other hand, if the trade bids unaggressively, he should do his best to win the bid by bidding more aggressively than the trade. Assuming $\hat{P}(\cdot)$ is a continuous function, the above implies that as Q increases $\hat{P}(Q)$ crosses the forty five degree line from above, and thus its slope at $\hat{P}(P^*) = P^*$ is less than unity; satisfying condition (24).

5. A Particular Model

In this section we generalize some of the results obtained by Rothkopf [9] for a one shot model to the case of sequential auctions. The work we refer to describes a model of a competitive auction in which there are n bidders, each with the same cost, c, of doing a job. It is assumed that each bidder independently makes an unbiased estimate of his cost and then multiplies his estimate by a factor P_i in order to arrive at his bid. The bidder's independent cost estimates are each assumed to come from a two parameter Weibull distribution with spread parameter m. Under these circumstances, the expected profit from this auction for bidder i is given by

$$E_i(Q_i,P_i) = cp_i \; (P_i p_i^{-1/m} - 1) \tag{25}$$

where

$$p_i = 1/[1 + (n-1)(P_i/Q_i)^m] \tag{26}$$

and

$$Q_i = \left[\frac{1}{(n-1)} \sum_{\substack{j=1 \\ j\neq i}}^{n} (1/P_j^m)\right]^{-1/m} \quad . \tag{27}$$

The quantity p_i in Eq. (25) is the probability that bidder i wins the auction while the term in the brackets times c represents his expected profit if he wins.

Suppose now that the one shot situation described above reoccurs at fixed time intervals and that each bidder has to consider the effect of his present bid on his future payoffs. In particular we shall assume as before that bidder i's objective is to maximize his present value of future rewards with discount factor D_i. Clearly the reward function (25) satisfies condition (1). We also assume the reaction function given by (20). Thus the identical bidders equilibrium policy can be derived from (23) and (24). This result in

$$P^* = \frac{m(n-1)n^{1/m}}{m(n-1)-1-F} \tag{28}$$

where F is defined as

$$F = \alpha D/(1-D) \quad . \tag{29}$$

For $F=0$, (28) reduces to the one shot equilibrium policy obtained in [9]. This is consistent with the observation made in Section 4, that for $D=0$ or $\alpha=0$ (23) reduces to the one shot equilibrium condition.

As F is increased, P^* increases until the equilibrium disintegrates at

$$F = m(n-1)-1 \tag{30}$$

Therefore we must restrict α to

$$\alpha < [m(n-1)-1](1-D)/D \ . \tag{31}$$

At equilibrium the expected value of the winning bid will be

$$c\,P^*\,n^{-1/m} = c[1 + \frac{1+F}{m(n-1)-1-F}] \ , \tag{32}$$

and the expected profit in each auction of each of the n bidders will be

$$E(P^*,P^*) = \frac{c\,P^*\,n^{-1/m} - c}{n} = \frac{c}{n} \cdot \frac{1+F}{m(n-1)-1-F} \ . \tag{33}$$

If the competition begins with equilibrium strategies, the present value of a bidder's profits in the present and all future auctions is given by

$$V(P^*) = E(P^*,P^*)/(1-D) \ . \tag{34}$$

Obviously, the bidders are better off at equilibrium if the trade reacts strongly to price cutting (i.e., if α is large) than if it does not. The degree of reaction that is likely to occur will probably depend on a number of institutional factors not fully represented by the model. These would certainly include the speed and certainty with which competitors can discern a policy change and the extent to which the competitors in one auction are likely to be the same as the competitors in the succeeding auctions.

6. Discussion and Conclusions

This paper accomplishes two things. First of all we have derived an equation for an optimal infinite horizon bidding strategy. This equation is quite general. It can be used for a wide variety of assumptions about how competitive policy will change in reaction to the bidding policy of one bidder and how a bidder's profit in an auction depends upon his policy and that of his competitors.

Secondly, we have applied this model to a particular generalization of a symetric, n bidder, game theoretic, one-shot bidding model. In doing so, we have changed the nature of the model. It is no longer strictly a game theoretic model. Each bidder is now assumed to be acting on a behavioral assumption about how his competitors will react in the future to his present bidding policy. This assumption may not be satisfied by the actual behavior of the bidders even if each bidder uses a similar model. By an appropriate choice of our parameter α, it may be possible to close the gap between what each bidder expects his competitors to do and what each competitor would do if he followed the advice of the model. This, however, has not yet been done.

In spite of the possible gap between expected reaction and actual reaction, we believe that the results given in equations (28), (32), and (33) provide insight into the effect of competitive reaction on optimal bidding policy and the profit to be paid by bid takers. It is interesting that the effect of the sequential nature of auctions depends only upon a factor that is the product of a parameter that measures the strength of competitive reaction and a simple function of the discount factor between auctions. It is also useful to observe that the expected profit of the bidders is quite nonlinearly dependent upon this factor. This suggests a number of tactics that bid takers can pursue if they suspect that they are paying excessive profits to suppliers due to tacit collusion. They may be able to increase m by reducing the uncertainty the bidders face, increase n by bringing in additional bidders, decrease D by taking bids less frequently, and decrease α by changing institutional factors. These steps might include making it more difficult for the trade to react by keeping the amount of the winning bid secret and by frequently changing the list of invited bidders so that there is usually at least one new bidder present.

For bidders the message of this model is restrain your aggressiveness in repetitive bidding situations if you think doing so will influence your competitors to behave less aggressively in the future. Also, a bidder should try to convince his competitors that he will react to their policy changes.

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